Effectiveness of the Global Modulus of Continuity on Metric Spaces

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Abstract. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. By definition, there is a function \(h : (f, x, \epsilon) \mapsto \delta, (\delta > 0)\), such that for all continuous function \(f : X \rightarrow Y, x \in X\) and \(\epsilon > 0\): \(\forall x' \in X (d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon)\). By a recent result of Repovš and Semenov [8], there is a function \(h\) continuous in \(f, x\) and \(\epsilon\) with this property, if \((X, d_X)\) is locally compact. Based on Weihrauch’s frameworks on computable metric space ([13]), we effectivize this result by showing that there is a computable function of this type. The proof is a direct construction not depending on [8].

Key words: Modulus of Continuity; Metric Space; Effective Analysis.

1 Introduction

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A function \(f : X \rightarrow Y\) is continuous if, for any \(x \in X\) and any \(\epsilon > 0\), there exists a \(\delta > 0\) such that

\[\forall x' \in X (d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon).\]  (1)

In other words, \(f\) is continuous if and only if there is a (total) function \(\tilde{\delta} : X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that, for any \((x, \epsilon) \in X \times \mathbb{R}^+\),

\[\forall x' \in X (d_X(x, x') < \tilde{\delta}(x, \epsilon) \implies d_Y(f(x), f(x')) < \epsilon).\]  (2)

The function \(\tilde{\delta}\) is called a modulus of continuity of \(f\).

The discussion about modulus of continuity is an interesting and important topic both in classical and effective analysis (see e.g., [2, 5, 10, 12]). For example,
Ko [2] shown that, if \( f : [a; b] \rightarrow \mathbb{R}^+ \) is a computable (hence continuous) real function, then there is a recursive function \( m : \omega \rightarrow \omega \) such that the function \( \delta \) defined by \( \delta(x, \epsilon) = 2^{-\lfloor 1/\epsilon \rfloor} \) is a modulus of (uniform) continuity of \( f \). For the classically locally compact metric space \( X \), Repovš and Semenov proved in [8] that, every continuous function \( f : X \rightarrow Y \) possesses a continuous modulus of continuity. In fact, they have proved even more that, it is possible to determine an appropriate \( \delta > 0 \) satisfying (1) as a continuous function of the triple \( (f, x, \epsilon) \) under proper topology. More precisely, let \( C(X, Y) \) be the set of all continuous functions from \( X \) into \( Y \), endowed with the topology of uniform convergence, i.e., the \( \epsilon \)-neighbourhood of \( f \in C(X, Y) \) is the set \( B(f, \epsilon) := \{ g \in C(X, Y) : \forall x \in X (d_Y(f(x), g(x)) < \epsilon) \}. \) Then the following has been proved in [8].

**Theorem (Repovš and Semenov [8]).** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces and suppose that \( X \) is locally compact. Then there exists a continuous function \( \delta : C(X, Y) \times X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that, for every triple \( (f, x, \epsilon) \in C(X, Y) \times X \times \mathbb{R}^+ \),

\[
\forall x' \in X \, (d_X(x, x') < \delta(f, x, \epsilon) \implies d_Y(f(x), f(x')) < \epsilon).
\]

The function \( \delta \) in above theorem is called *global modulus of continuity* of function space \( C(X, Y) \). The proof of Theorem applies Michael's Selection Theorem (cf. [6]). That is, the function \( \delta \) exists as a selection function of some lower semi continuous set valued function. Thus it is quite ineffective.

Our main purpose of this paper is to discuss the effectiveness about the global modulus of continuity of \( C(X, Y) \) when \( X \) is (effectively) locally compact. It is, of course, only possible after a computability framework about metric spaces is well established. In [13], Weihrauch introduced computability on metric space by the representation theory. In this theory, computability on finite and infinite sequences over some finite alphabet of symbols are defined explicitly, e.g., by Turing machines, and computability on other sets are introduced by representations, i.e., naming systems, where infinite sequences of symbols are used as names (cf, e.g. [3, 11, 12]). In this paper we will work in this framework and prove finally a computationally effective version of the above theorem, i.e. there is a computable global modulus of continuity of \( C(X, Y) \). Thus, we can choose effectively a positive real \( \delta \) satisfying (1) from any continuous function \( f : X \rightarrow Y \), element \( x \in X \) and positive real \( \epsilon \).

We fix \( \Sigma \) to be a finite alphabet containing all symbols we need. \( \Sigma^* \) and \( \Sigma^\omega \) are the sets of all finite words and infinite sequences over \( \Sigma \), respectively. The computability theory on \( \Sigma^* \) and \( \Sigma^\omega \) have been well established ([9, 12]). To discuss the computability on other sets, we need their representations by the elements of \( \Sigma^* \) or \( \Sigma^\omega \). For any set \( A \) with the cardinality of at most continuum, a representation of \( A \) is simply a surjective function \( \delta : \Sigma^a \rightarrow A \) for \( a \in \{*, \omega \} \). For example, let \( \Sigma = \{0, 1, \#, \!, \} \), the functions \( \nu_N : \Sigma^* \rightarrow \omega \), \( \nu_Q : \Sigma^* \rightarrow \mathbb{Q} \) and \( \rho_R : \Sigma^\omega \rightarrow \mathbb{R} \) be defined respectively by

\[
\nu_N(w) = n \iff w = 1^n;
\]