Attractors of $D$-dimensional Linear Cellular Automata

Giovanni Manzini$^{1,2}$, Luciano Margara$^{3,4}$

$1$ Dipartimento di Scienze e Tecnologie Avanzate, Università di Torino, Via Cavour 84, 15100 Alessandria, Italy. manzini@mfna1.unipmn.it

$2$ Istituto di Matematica Computazionale, Via S. Maria, 46, 56126 Pisa, Italy.

$3$ Dipartimento di Scienze dell’Informazione, Università di Bologna, Mura Anteo Zamboni 7, 40127 Bologna, Italy. margara@cs.unibo.it

$4$ International Computer Science Institute (ICSI), Berkeley CA.

Abstract. In this paper we study the asymptotic behavior of $D$-dimensional linear cellular automata over the ring $\mathbb{Z}_m$ ($D \geq 1$, $m \geq 2$). In the first part of the paper we consider non-surjective cellular automata. We prove that, after a transient phase of length at most $\lfloor \log_2 m \rfloor$, the evolution of a linear non-surjective cellular automata $F$ takes place completely within a subspace $Y_F$. This result suggests that we can get valuable information on the long term behavior of $F$ by studying its properties when restricted to $Y_F$. We prove that such study is possible by showing that the system $(Y_F, F)$ is topologically conjugated to a linear cellular automata $F^*$ defined over a different ring $\mathbb{Z}_{m^*}$. In the second part of the paper, we study the attractor sets of linear cellular automata. Recently, Kurka [8] has shown that CA can be partitioned into five disjoint classes according to the structure of their attractors. We present a procedure for deciding the membership in Kurka’s classes for any linear cellular automata. Our procedure requires only gcd computations involving the coefficients of the local rule associated to the cellular automata.

1 Introduction

Cellular Automata (CA) are dynamical systems consisting of a $D$-dimensional lattice of variables which can take a finite number of discrete values. The global state of the CA, specified by the values of all the variables at a given time, evolves in synchronous discrete time steps according to a given local rule which acts on the value of each single variable. For an introduction to the CA theory and an extensive and up-to-date bibliography see [5].

In this paper we restrict our attention to the class of linear CA (CA based on a linear local rule defined over the ring $\mathbb{Z}_m$). Despite of their simplicity that makes it possible a detailed algebraic analysis, linear CA exhibit many of the complex features of general CA. Recently, many important properties of linear CA have been completely characterized (see Fig. 1). These properties have been introduced for the study of discrete time dynamical systems (see for example [4]). Among other things, they provide valuable information on the long
Fig. 1. Characterization of set theoretic and topological properties of linear CA over \( \mathbb{Z}_m \) in terms of the coefficients \( \lambda_i \)'s (for \( D \)-dimensional CA) or \( a_i \)'s (for 1-dimensional CA). \( P \) denotes the set of prime factors of \( m \).

The term behavior of a complex system. The results mentioned above show that it is often possible to make a detailed analysis of the global dynamical behavior of a linear CA by analyzing the coefficients of its local rule.

In the first part of this paper we study the asymptotic behavior of non-surjective linear CA. By looking at Fig. 1 we notice that a non-surjective CA cannot be neither ergodic, nor (strongly) transitive, nor (positively) expansive, nor regular. This makes most of the above listed results inapplicable to the study of the long term behavior of non-surjective linear CA over \( \mathbb{Z}_m \). However, we prove (Theorem 5) that for any non-surjective linear CA \( F \), there exists a subspace \( Y_F \) such that, for any configuration \( x \), \( F^k(x) \in Y_F \) for all \( k \geq \lfloor \log_2 m \rfloor \). That is, after a transient phase of length at most \( \lfloor \log_2 m \rfloor \), the evolution of the system takes place completely within the subspace \( Y_F \). This result indicates that, in order to study the asymptotic behavior of non-surjective linear CA, one should analyze the behavior of the map \( F \) over the subspace \( Y_F \). We show how to carry out this analysis by proving (Theorem 6) that the behavior of \( F \) over \( Y_F \) is identical to the behavior of a linear surjective map \( F^* \) defined over a configuration space isomorphic to \( Y_F \). We also give an explicit formula for the coefficients of the local rule associated to \( F^* \). The knowledge of these coefficients makes it possible to easily recognize if the map \( F \) restricted to \( Y_F \) satisfies any of the properties reported in Fig. 1.

In the second part of this paper, we make a further step in the analysis of the long term behavior of linear CA by studying the structure of their attractors. Informally, an attractor for a dynamical system \((X,F)\) is a subset \( Z \subseteq X \) of configurations such that the forward trajectory under iterations of \( F \) of any configuration which is sufficiently close to \( Z \) gets closer and closer to \( Z \). Attractors of CA have been studied for example by Hurley [6], Kurka [8], and Blanchard et al. [1]. In particular, Kurka partitions the set of CA into five disjoint classes, labeled \( A_1 - A_5 \), according to the structure of their attractors. We prove that for