Abstract. Building on work of Gaifman [Gai82] it is shown that every first-order formula is logically equivalent to a formula of the form \( \exists x_1, \ldots, x_t \forall y \varphi \) where \( \varphi \) is \( r \)-local around \( y \), i.e. quantification in \( \varphi \) is restricted to elements of the universe of distance at most \( r \) from \( y \).

From this and related normal forms, variants of the Ehrenfeucht game for first-order and existential monadic second-order logic are developed that restrict the possible strategies for the spoiler, one of the two players. This makes proofs of the existence of a winning strategy for the duplicator, the other player, easier and can thus simplify inexpressibility proofs.

As another application, automata models are defined that have, on arbitrary classes of relational structures, exactly the expressive power of first-order logic and existential monadic second-order logic, respectively.

1 Introduction

First-order (FO) logic and its extensions play an important role in many branches of (theoretical) computer science. Examples that will be considered in this paper are automata theory and descriptive complexity. Since Büchi's and Elgot's famous characterization of the regular string languages as the sets of models of (existential) monadic second-order (MSO) sentences, (existential) MSO logic has been used as a guideline in the search for reasonable automata models for other kinds of structures like trees or graphs. In descriptive complexity, since Fagin [Fag74] showed that the complexity class \( \mathbf{NP} \) coincides with the sets of models of existential second-order (\( \Sigma^1_1 \)) sentences, many complexity classes have been characterized by extensions of FO logic (for references, see [EF95]) and there is still hope that separations of complexity classes might be possible by separating the expressive power of the respective logics. For a recent result in this direction see the paper of Libkin and Wong in this volume.

Despite its importance as an ingredient for more expressive logics, it is well-known that the expressive power of FO logic is rather limited. It can only express properties that depend on the local appearance of a structure. This intuition has been formalized in different ways by Hanf [Han65] and Gaifman [Gai82]. Hanf showed that, for every first-order formula \( \psi \), there is an \( r \) such that whether \( \psi \) holds in a structure \( A \) ("\( A \models \psi \)"") only depends on the multiset of isomorphism types of all \( r \)-spheres in \( A \). Here an \( r \)-sphere is a substructure of \( A \) which is induced by all elements of \( A \) that have distance at most \( r \) from a fixed element.
of $A$. On the other hand, Gaifman showed that, for every first-order formula $\psi$, there are $r$ and $d$ such that whether $A \models \psi$ holds depends only on how many elements with pairwise disjoint $r$-neighbourhoods exist that fulfil $\theta$, for every formula $\theta$ of quantifier depth at most $d$.

The starting question for the present investigation was to which extent Hanf's and Gaifman's conditions could be combined. The goal was to replace the isomorphism types in Hanf's condition by something weaker and to get rid of the "disjoint $r$-neighbourhoods" constraint in Gaifman's condition. (For very interesting recent results concerning Hanf's and Gaifman's theorems from a different point of view see [Lib97,DLW97].) It is easy to see that the straightforward attempt to replace the isomorphism type of a sphere $S$ in Hanf's condition by its Hintikka type for some $d$ (i.e. by the set of formulas of quantifier depth at most $d$ that hold in $S$) does not work. A counterexample is given by the set of clique graphs. For every $d$, the spheres of a graph consisting of one $2d$-clique fulfill exactly the same formulas of quantifier depth at most $d$ as those of a graph which consists of two disjoint $d$-cliques. Nevertheless, it turns out that it is indeed possible to combine the two approaches in the following sense. For every FO formula $\psi$ there are $l$ and $r$ such that $A \models \psi$ if and only if it is possible to put $l$ pebbles onto $A$ such that in the resulting structure all $r$-spheres fulfil the same FO formula $\varphi$. Put in another way, every FO formula is logically equivalent to a formula of the form $\exists x_1, \ldots, x_l \forall y \varphi$ where $\varphi$ is $r$-local around $y$, i.e. quantification in $\varphi$ is restricted to elements of the universe with distance at most $r$ from $y$. From this normal form one can easily derive normal forms for other logics like monadic second-order logic. For existential monadic second-order logic we can show even more restricted normal forms.

As one application of the normal forms we get variants of the Ehrenfeucht game [Ehr61] for first-order logic and existential monadic second-order logic in which the spoiler has only restricted global access to the structures that are played. Another application concerns a form of automata on relational structures. It is well-known that regular sets of strings can be obtained as projections of locally testable sets, namely the sets of transition sequences of a (nondeterministic) automaton. Thomas [Tho91] used this idea to extend the notion of recognizability to other objects like grids and graphs of uniformly bounded degree. The normal forms allow to generalize the method of local testing further to sets of arbitrary relational structures. Moreover, they maintain the correspondence to definability in a natural logic.

In Section 2 we give basic definitions and fix some notation. In Section 3 we show the normal form theorems. In Section 4 we define the simplified games for first-order logic and monadic $\Sigma_1^*$-logic. The analogous results for automata are presented in Section 5. Section 6 contains a conclusion.

## 2 Definitions and Notations

A relational signature $\sigma$ is a finite set of relation symbols $R$, each with a fixed arity $\alpha(R)$, and constant symbols $c$. We do not use function symbols. A $\sigma$-structure