Muller automata and bi-infinite words

by

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Abstract: We prove that the family \( \text{Rec}(\omega A^\omega) \) of regular sets of bi-infinite words is equal to the family of sets recognized by a deterministic Muller automaton. That extends a theorem of McNaughton for one-sided infinite words to the case of bi-infinite words.

1. Introduction

There exist two different definitions for a one-sided infinite word to be accepted by a finite automaton. These conditions are named the Muller [5] and Büchi [3] conditions. It is well known [4] that these conditions are equivalent.

The set of infinite words over a given alphabet \( A \), recognized by a finite automaton in either of the two acceptance conditions are the same. They are called the recognizable subsets of \( A^\omega \) and their set will be denoted by \( \text{Rec}(A^\omega) \).

However, when restricted to deterministic automata, the two conditions for acceptance are no longer equivalent. Each of these two families admits several characterization. The family of languages recognized by deterministic Büchi automata is the family of limits of regular sets of \( A^\omega \), denoted by \( \text{Rec}(A^\omega) \). On the other hand, McNaughton [5] has shown that:

1) the family of sets recognized by deterministic Muller automaton is the whole family \( \text{Rec}(\omega A^\omega) \). This means that every recognizable set of one-sided infinite words can be recognized by a deterministic Muller automaton. In the case of one-sided infinite words, it is not difficult to see that this property is equivalent to the fact that

2) \( \text{Rec}(A^\omega) \) is the Boolean closure of \( \text{Rec}(A^\omega) \).

We will prove in this paper the analogue of the property (1) for bi-infinite words, namely:

Theorem: The family of sets of bi-infinite words recognized by deterministic Muller automata is the whole family \( \text{Rec}(\omega A^\omega) \) of recognizable sets of bi-infinite words.

In the case of one-sided infinite words, property (1) is easily derived from property (2), by using the fact that the complement of a regular set recognized by a deterministic Büchi automaton is recognized by a deterministic Muller automaton.

This property is quite difficult to prove in the bi-infinite case, and in fact, it is a consequence of our theorem. Therefore, we cannot proceed along the lines of the proofs for the one-sided infinite case, and we shall give a direct proof of the theorem.

Observe that the automaton constructed in the proof of the theorem is a rather special: the condition of left acceptance is of Büchi type; however the full power of Muller acceptance is required on the right. There still remains an open problem: give an easy method for the construction of a deterministic Muller automaton recognizing the complement of a set recognized by a deterministic Muller automaton.
2. Preliminaries

Let $A$ be a finite alphabet. $A^*$ is the set of finite words on the alphabet $A$. Empty word is denoted by $\varepsilon$ and $A^+ = A^* \setminus \{\varepsilon\}$.

The set of right infinite words $\alpha = \alpha_0\alpha_1\ldots \alpha_n\ldots$ ($\alpha_n \in A$) is denoted by $A^\omega$. $\omega A$ is the set of left infinite words $\alpha = \ldots \alpha_n \alpha_{n-1} \alpha_0$. A bi-infinite word is an equivalence class in the set $A^\omega$ for the shift relation. The set of bi-infinite words is denoted by $\omega A^\omega$, and a bi-infinite word is written as one of its elements

$$\alpha = \alpha_n \ldots \alpha_1 \alpha_0 \alpha_1 \ldots \alpha_p \ldots \quad (\alpha_i \in A)$$

Let $X, Y, Z \subseteq A^\ast$.

We define $XY^\omega = \{u \in A^\omega/ u = xy_1\ldots y_n\ldots \text{ with } x \in X, y_n \in Y\}$

$\omega XYZ = \{u \in \omega A^\omega/ u = \ldots x_n \ldots x_1 yz_1\ldots z_n\ldots \text{ with } x_n \in x, y \in Y, z_n \in Z\}$

A Buchi finite automaton is a 4-tuple $A = (Q, F, I, T)$ such that:

- $Q$ is a finite set of states
- $I \subseteq Q$ is the set of transitions
- $T$ is the set of initial (resp. final) states.

The finite behaviour of $A$, denoted by $A^*\subseteq A^*$, is the set of the labels $u$ of paths starting in $I$ and ending in $T$; we write $I \longrightarrow T$.

The right infinite behaviour of $A$, $A^\omega$ is the set of labels $u$ of infinite paths which start in $I$, and pass infinitely often through $T$.

We write $I \longrightarrow \omega T$. In a dual way, $\omega A$ is the set of left infinite words $u$ which are label of paths passing infinitely often through $I$ and ending in $T$. We write $I \longrightarrow \omega T$.

At last, $\omega A^\omega$ is the set of bi-infinite words $u$, labels of bi-infinite paths passing infinitely often in $I$ on the left, an infinitely often in $T$ on the right. We write $I \longrightarrow \omega T$.

The automaton $A$ is said to be deterministic if, for every $q, q', q'' \in Q$ and every $a \in A$:

$$(q, a, q') \in F \quad \text{and} \quad (q, a, q'') \in F \implies q' = q''$$

A subset $L \subseteq A^\ast$ is recognizable if it is the finite behaviour of a finite automaton. $\text{Rec}(A^\ast)$ is the family of recognizable sets of $A^\ast$.

In the same way we define the families $\text{Rec}(A^\omega)$, $\text{Rec}(\omega A)$ and $\text{Rec}(\omega A^\omega)$.

The two following propositions characterize $\text{Rec}(A^\omega)$ and $\text{Rec}(\omega A^\omega)$. 