PRODUCTS OF GROUP LANGUAGES

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Concatenation product is, together with Kleene's star operation, one of the most fascinating operations on recognizable (= rational, regular) languages. The study of this operation produced numerous fundamental results like Schützenberger's theorem on star-free languages, Brzozowski's results on the dot depth hierarchy, Simon's theorem on piecewise testable languages, Straubing's characterization of varieties closed under product, results of Mc Naughton and Thomas on the connexion with first order logic, etc. It also had a considerable influence on the rest of the theory and many algebraic tools were originally introduced to produce better proofs of old or new results.

In fact, it appears that the truly fundamental operation is not exactly the usual concatenation but a variant of it, that consists to associate to languages $L_0, \ldots, L_n$ the language $L_0 a_1 L_1 \ldots a_n L_n$ where $a_1, \ldots, a_n$ are given letters of the alphabet. Notice that this operation is not mysterious at all. It is used for instance to obtain a rational expression associated to a finite automaton in the classical algorithm of Mc Naughton and Yamada. Therefore, in this paper, the term "product" will refer to this variant of concatenation product.

With this operation in hand, it is not difficult to construct hierarchies of recognizable languages. Start with a boolean algebra of languages : this will be the level 0 of your hierarchy. Then define level $n+1$ as the boolean algebra generated by products (in the new sense) of languages of level $n$. If you start with the trivial boolean algebra $\{\emptyset, A^*\}$ you obtain Straubing's hierarchy. If you start with endwise testable (or "generalized definite") languages, you get Brzozowski's hierarchy, also called dot-depth hierarchy.

The aim of this paper is to study the hierarchy whose level 0 consists of all group languages. In this case, the union of all levels of the hierarchy is the closure of group languages under product and boolean operations. Our first result
shows that \( \mathcal{U} \) is a decidable variety of languages. That is, given a recognizable language \( L \), one can decide whether \( L \) belongs to \( \mathcal{U} \) or not. Our second result states that our hierarchy is strict. In fact, this result still holds if one takes as level 0 an arbitrary subvariety of the variety of group languages.

The rest of the paper is devoted to the study of level 1. It turns out that this variety of languages, and the corresponding variety of monoids \( \mathcal{O}_G \), appear in many different contexts. First, \( \mathcal{O}_G \) is exactly the variety \( J \times \mathcal{G} \) generated by all semidirect products of a \( J \)-trivial monoid by a group. This result is interesting because \( J \) is also the first level of Straubing's hierarchy \( V \). Thus, at least for levels 0 and 1, the operation \( \mathcal{V} \rightarrow \mathcal{V} \star \mathcal{G} \) is the bridge between Straubing's hierarchy and our hierarchy. Similarly, it is known [14] that the operation \( \mathcal{V} \rightarrow \mathcal{V} \star \mathcal{L} \) is the bridge between Straubing's hierarchy and Brzozowski's hierarchy (\( \mathcal{L} \) denotes the first level of Brzozowski's hierarchy). Second \( \mathcal{O}_G \) is also the variety generated by powermonoids of groups. In fact we first prove the language counterpart of this result: a language has level \( \leq 1 \) in our hierarchy iff it belongs to the boolean algebra generated by all languages of the form \( L^q \) where \( L \) is a group language and \( q \) is a length preserving morphism (strictly alphabetic, letter to letter morphism).

Finally, languages of level 1 arise in the study of the finite group topology for the free monoid \( \mathcal{G}_3 \).

An important problem is to know whether the first level of our hierarchy is a decidable variety. The answer is positive in the case of Straubing's hierarchy and Brzozowski's hierarchy, and thus we hope a positive answer also in our case. The discussion of this problem motivated the introduction of a new variety of monoids, denoted by \( \mathcal{B}_G \). \( \mathcal{B}_G \) is the variety of all monoids \( M \) such that, for every idempotent \( e,f \in M \), \( efe = e \) implies \( e = f \). Several equivalent definitions are given in the paper. In particular, we show that a monoid \( M \) is in \( \mathcal{B}_G \) iff the submonoid generated by all idempotents of \( M \) belongs to \( J \). We also prove that \( \mathcal{B}_G \) is generated by all monoids that are, in some sense, extensions of a group by a monoid of \( J \). Finally, \( \mathcal{O}_G \) is contained in \( \mathcal{B}_G \) but we don't know if this inclusion is strict or not. If \( \mathcal{O}_G = \mathcal{B}_G \),