Convexity of Minimal Total Dominating Functions in Graphs

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Abstract. A total dominating function (TDF) of a graph $G = (V, E)$ is a function $f : V \rightarrow [0, 1]$ such that for each $v \in V$, the sum of $f$ values over all neighbours of $v$ (i.e., all vertices adjacent to $v$) is at least one. Integer-valued TDFs are precisely the characteristic functions of total dominating sets of $G$. A minimal TDF (MTDF) is one such that decreasing any value of it makes it non-TDF. An MTDF $f$ is called universal if convex combinations of $f$ and any other MTDF are minimal. We give a sufficient condition for an MTDF to be universal which generalises previous results. Also we define a splitting operation on a graph $G$ as follows: take any vertex $v$ in $G$ and a vertex $w$ not in $G$ and join $w$ with all the neighbours of $v$. A graph $G$ has a universal MTDF if and only if the graph obtained by splitting $G$ has a universal MTDF. A corollary is that graphs obtained by the operation from paths, cycles, complete graphs, wheels, and caterpillar graphs have a universal MTDF.

1 Introduction

A total dominating function (TDF) of a graph $G = (V, E)$ is a non-negative real function $f$ from the vertex set $V$ to the unit interval $[0, 1]$ such that for each $v \in V$, the sum of $f$ values over all neighbours of $v$ is at least one. That is $f(N(v)) \geq 1$, where $N(v)$ is the set of neighbours of $v$, $N(v) = \{w \in V | vw \in E\}$ and $f(N(v))$ denotes $\sum_{u \in N(v)} f(u)$. Zero-one valued TDFs are precisely the characteristic functions of total dominating sets of $G$, where a subset $X$ of vertices is called a total dominating set of $G$ if every $v \in V$ has a neighbour in $X$. The reader is referred to [9] for an excellent bibliography concerning domination in graphs. Total dominating functions and total dominating sets have been studied in [2-9]. In particular, the convexity of minimal total dominating functions has been examined [6, 7, 10, 5]. A minimal total dominating function (MTDF) $f$ is a TDF such that $f$ does not remain a TDF if any $f(v), v \in V$ is decreased. It is easily shown that a convex combination of TDFs is still a TDF. But, a convex combination of MTDFs is not necessarily an MTDF. Figure 1 shows two MTDFs $f$ and $g$ such that a convex combination of $f$ and $g$, say $\frac{1}{3}f + \frac{1}{2}g$, is not an MTDF. This is easily verified using Theorem 2.
Motivated by this fact, [6] introduced the notion of a universal MTDF. An MTDF is called universal if any convex combination of \(g\) and any other MTDF is also an MTDF. Another reason for the study of convexity and universal MTDFs is the following interpolation problem due to Hedetniemi [8] (the interpolation problem in [8] concerns dominating functions [6]). The aggregate of a TDF \(f\) is the sum of \(f\) values over all vertices. Suppose that we are given MTDFs \(f\) and \(g\) with aggregates \(\alpha\) and \(\beta\), and a number \(t \in (\alpha, \beta)\). Does \(G\) have an MTDF with aggregate \(t\)? It is easily seen that the answer is "yes" provided \(G\) has a universal MTDF.

In Section 2, we give a sufficient condition for an MTDF to be universal, which generalizes previous results due to Cockayne et al [6]. In Section 3, we present an operation on graphs such that a graph \(G\) has a universal MTDF if and only if the graph obtained by applying the operation has a universal MTDF. This operation gives several classes of graphs that have universal MTDFs. Finally, we give some open problems.

2 A sufficient condition for an MTDF to be universal

In this section, we introduce two types of vertices, dominating vertices and low vertices. In a tree, dominating vertices and low vertices are equivalent to so-called short vertices and hot vertices defined in [6], respectively. Our main theorem, Theorem 6, gives a sufficient condition for an MTDF to be universal. This generalizes one of the main theorems (Theorem 18) in [6]. If we restrict the graph in Theorem 6 to be a tree, then the condition in our theorem is necessary and sufficient, see [5].

For a TDF \(f\), we define \(B_f\), the boundary of \(f\), to be the set \(\{v \in V | f(N(v)) = 1\}\), and \(P_f\), the positive set of \(f\), to be the set \(\{v \in V | f(v) > 0\}\). For subsets \(A, B\) of \(V\), we write \(A \rightarrow B\) if every vertex in \(B\) has a neighbour in \(A\), i.e., \(B\) is a subset of \(\bigcup_{v \in A} N(v)\).

We need the following two theorems, Theorem 1 and Theorem 2, from [6].