Convexity of Minimal Total Dominating Functions in Graphs

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Abstract. A total dominating function (TDF) of a graph $G = (V, E)$ is a function $f : V \to [0, 1]$ such that for each $v \in V$, the sum of $f$ values over all neighbours of $v$ (i.e., all vertices adjacent to $v$) is at least one. Integer-valued TDFs are precisely the characteristic functions of total dominating sets of $G$. A minimal TDF (MTDF) is one such that decreasing any value of it makes it non-TDF. An MTDF $f$ is called universal if convex combinations of $f$ and any other MTDF are minimal. We give a sufficient condition for an MTDF to be universal which generalises previous results. Also we define a splitting operation on a graph $G$ as follows: take any vertex $v$ in $G$ and a vertex $w$ not in $G$ and join $w$ with all the neighbours of $v$. A graph $G$ has a universal MTDF if and only if the graph obtained by splitting $G$ has a universal MTDF. A corollary is that graphs obtained by the operation from paths, cycles, complete graphs, wheels, and caterpillar graphs have a universal MTDF.

1 Introduction

A total dominating function (TDF) of a graph $G = (V, E)$ is a non-negative real function $f$ from the vertex set $V$ to the unit interval $[0, 1]$ such that for each $v \in V$, the sum of $f$ values over all neighbours of $v$ is at least one. That is $f(N(v)) \geq 1$, where $N(v)$ is the set of neighbours of $v$, $N(v) = \{w \in V \mid vw \in E\}$ and $f(N(v))$ denotes $\sum_{u \in N(v)} f(u)$. Zero-one valued TDFs are precisely the characteristic functions of total dominating sets of $G$, where a subset $X$ of vertices is called a total dominating set of $G$ if every $v \in V$ has a neighbour in $X$. The reader is referred to [9] for an excellent bibliography concerning domination in graphs. Total dominating functions and total dominating sets have been studied in [2-9]. In particular, the convexity of minimal total dominating functions has been examined [6, 7, 10, 5]. A minimal total dominating function (MTDF) $f$ is a TDF such that $f$ does not remain a TDF if any $f(v), v \in V$ is decreased. It is easily shown that a convex combination of TDFs is still a TDF. But, a convex combination of MTDFs is not necessarily an MTDF. Figure 1 shows two MTDFs $f$ and $g$ such that a convex combination of $f$ and $g$, say $\frac{1}{2}f + \frac{1}{2}g$, is not an MTDF. This is easily verified using Theorem 2.
Motivated by this fact, [6] introduced the notion of a universal MTDF. An MTDF is called universal if any convex combination of \( g \) and any other MTDF is also an MTDF. Another reason for the study of convexity and universal MTDFs is the following interpolation problem due to Hedetniemi [8] (the interpolation problem in [8] concerns dominating functions [6]). The aggregate of a TDF \( f \) is the sum of \( f \) values over all vertices. Suppose that we are given MTDFs \( f \) and \( g \) with aggregates \( \alpha \) and \( \beta \), and a number \( t \in (\alpha, \beta) \). Does \( G \) have an MTDF with aggregate \( t \)? It is easily seen that the answer is "yes" provided \( G \) has a universal MTDF.

In Section 2, we give a sufficient condition for an MTDF to be universal, which generalizes previous results due to Cockayne et al [6]. In Section 3, we present an operation on graphs such that a graph \( G \) has a universal MTDF if and only if the graph obtained by applying the operation has a universal MTDF. This operation gives several classes of graphs that have universal MTDFs. Finally, we give some open problems.

## 2 A sufficient condition for an MTDF to be universal

In this section, we introduce two types of vertices, dominating vertices and low vertices. In a tree, dominating vertices and low vertices are equivalent to so-called short vertices and hot vertices defined in [6], respectively. Our main theorem, Theorem 6, gives a sufficient condition for an MTDF to be universal. This generalizes one of the main theorems (Theorem 18) in [6]. If we restrict the graph in Theorem 6 to be a tree, then the condition in our theorem is necessary and sufficient, see [5].

For a TDF \( f \), we define \( B_f \), the boundary of \( f \), to be the set \( \{ v \in V | f(N(v)) = 1 \} \), and \( P_f \), the positive set of \( f \), to be the set \( \{ v \in V | f(v) > 0 \} \). For subsets \( A, B \) of \( V \), we write \( A \rightarrow B \) if every vertex in \( B \) has a neighbour in \( A \), i.e., \( B \) is a subset of \( \bigcup_{v \in A} N(v) \).

We need the following two theorems, Theorem 1 and Theorem 2, from [6].