Precise Estimation of the Order of Local Testability of a Deterministic Finite Automaton

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Abstract. A locally testable language $L$ is a language with the property that for some nonnegative integer $k$, called the order or the level of local testability, whether or not a word $u$ in the language $L$ depends on (1) the prefix and suffix of the word $u$ of length $k - 1$ and (2) the set of intermediate substrings of length $k$ of the word $u$. For given $k$ the language is called $k$-testable.

We give necessary and sufficient conditions for the language of an automaton to be $k$-testable in the terms of the length of paths of a related graph. Some estimations of the upper and of the lower bound of order of testability follow from these results.

We improve the upper bound on the order of testability of locally testable deterministic finite automaton with $n$ states to $\frac{n^2}{2} + 1$. This bound is the best possible.

We give an answer on the following conjecture of Kim, McNaughton and McCloskey for deterministic finite locally testable automaton with $n$ states: “Is the order of local testability no greater than $\Omega(n^{1.5})$ when the alphabet size is two?”

Our answer is negative. In the case of size two the situation is the same as in general case: the order of local testability is $\Omega(n^2)$.

1 Introduction

The concept of local testability was first introduced by McNaughton and Papert [9] and since then has been extensively investigated from different points of view [1, 3, 4, 6, 8, 11, 12, 14, 15, 16]. This concept is connected with languages, finite automata and semigroups. In [10], local testability is discussed in terms of “diameter-limited perceptrons”. Locally testable languages are a generalization of the definite and reverse-definite languages, which can be found, for example, in [2, 13].

In [5] necessary and sufficient conditions for an automaton to be locally testable were found. In [6] the NP-hardness of finding of the order of local testability was proved. The necessary and sufficient conditions of $k$-testability in the terms of 5-tuple graph were found in [6]. An estimation for the order of local testability for an arbitrary deterministic finite automaton was found first in [5]
and then improved in [6]. The upper bound from [6] is $2n^2 + 1$, where $n$ is the number of states of the automaton.

For the state transition graph $\Gamma$ of an automaton we consider some subgraphs of the direct product $\Gamma \times \Gamma$. We introduce in this paper sufficient and necessary conditions for the automaton and transition semigroup of the automaton to be $k$-testable in terms of the length of some paths without loops on these graphs. This gives us some upper and some lower bounds on the order of local testability.

In the case that the state transition graph is strongly connected the sufficient conditions are necessary as well and algorithm of finding of the level of local testability is polynomial and not NP-hard as in the general case [6].

As corollary we receive the precise upper bound on the order of local testability for deterministic finite locally testable reduced automaton with $n$ states. It is equal to $(n^2 - n)/2 + 1$. This result improves the estimations from [5, 6] and finishes investigations in this direction.

In [4, 6] one can find conjecture that in the case of the alphabet two the upper bound on the order of local testability for the deterministic finite locally testable reduced automaton with $n$ states is not greater than $\Omega(n^{1.5})$.

We consider in this paper an example of sequence of deterministic finite automata with $n$ states whose alphabet size is two. It will be proved that the considered automata are locally testable and their order of local testability is $\Omega(n^2)$. So the problem from [4, 6] is solved negatively.

Our example is one between examples of locally testable automata whose order of testability is greater than the number of its states. First such astonishing example of an automaton with 28 states had appeared in [4, 6]. (Note that the order of testability of the considered automaton found in these papers is not correct. It is more greater than 126 [4] or 127 [6]. The conjuncture of the authors that the automaton has the maximal order of testability for automaton with 28 states and alphabet size two is not correct too. There exist a deterministic finite 142-testable automaton with 28 states and alphabet size two).

The description of the identities of $k$-testable semigroup from [14] is used here. The concept of the graph is inspired by the works [4, 5] of Kim, McNaughton and McCloskey. The purely algebraic approach proved to be fruitful [11, 14, 15] and in this paper we use this technique too.

2 Notation and definitions

Let $\Sigma$ be an alphabet and let $\Sigma^+$ denote the free semigroup on $\Sigma$. If $w \in \Sigma^+$, let $|w|$ denote the length of $w$. Let $k$ be a positive integer. Let $i_k(w)$/$t_k(w)$ denote the prefix [suffix] of $w$ of length $k$ or $w$ if $|w| < k$. Let $F_k(w)$ denote the set of factors of $w$ of length $k$. A language $L$ [a semigroup $S$] is called $k$-testable if there is an alphabet $\Sigma$ [and a surjective morphism $\phi : \Sigma^+ \rightarrow S$] such that for all $u$, $v \in \Sigma^+$, if $i_{k-1}(u) = i_{k-1}(v), t_{k-1}(u) = t_{k-1}(v)$ and $F_k(u) = F_k(v)$, then either both $u$ and $v$ are in $L$ or neither is in $L$ [$u\phi = v\phi$].

This definition follows [1, 4]. In [9] the definition differs by considering prefixes and suffixes of length $k$. 