INTRODUCTION

Why study linear difference equation?

Let \( L := \sum_{i=0}^{n} a_i \delta^i \)

be a linear difference operator with polynomial coefficients \((a_i \in k[x])\) and \( \delta \) the operator of translation:

\( \delta u(x) = u(x-1) \)

In many domains, linear difference equations are of great importance. And this is fundamental in, at least, two ways:

1st - the study of asymptotic solution of linear differential equations in the neighbourhood of irregular singularities ([1], [2]).

2nd - the use of these relations to compute the previous solutions (for example to generate the Bessel functions \( J_n \) of 1st order...).

Even if the study of such equations started long time ago, with Leonard Euler and his work on the \( \Gamma \) function solution of the equation \( u(x+1) = x \ u(x) \); we have not yet any satisfactory theory at our disposal, not even concerning the notion of solutions (cf. Ramis [4]). The algorithmic studies are also very poor.

The study of asymptotic solutions of \( L \) may be undertaken in two ways:

- a generalization of the method of Galbrun [5], Poincaré [6], and especially Birkhoff [7]. This study has been undertaken by Duval [12] and Loday [13].

- a suitable use of an operational method of Boole [10] which leads, more easily than that of Birkhoff, to an algorithmic treatment. At the present time, however this study is not yet sufficient to take into account all the degenerate solutions.

In this paper, we present a study of the second approach which will contain:

1) The \( \gamma \) and \( \rho \) operators of Boole.
2) The Boole-Probenius method.
3) The Newton polygon of linear difference equation and a classification.
4) Algorithms.
5) Conclusion.
1) THE AND \( \pi \) OPERATORS OF BOOLE

1.1 - Definition of these operators

The two fundamental operators of this theory are the following:

\[
\pi u(x) = x(u(x) - u(x-1))
\]

\[
\rho u(x) = \frac{\Gamma(x+1)}{\Gamma(x)} u(x-1)
\]

They have the following properties:

(P1): If \( m \) is a positive integer

\[
(\pi+m)^m u(x) = x^m u(x)
\]

\[
\rho^m u(x) = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} u(x-m)
\]

If \( \rho^m \) is applied to the function identically equals to 1 we get:

\[
\rho^m 1 = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} (x-m+1)
\]

The following theorem is of great importance in applications.

**Theorem 1:**

If \( P \) is an element of \( k[x] \)

\[
P(\pi)^m = \rho^m P(\pi+m)
\]

In particular, if \( \rho^m 1 = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} \) we see that \( P(\pi)^m = P(m)^n \).

The fundamental idea of the Boole method is to replace the 2 operators (multiplication by \( x \) and the operator \( \delta \) of translation) which define \( L \), by the two operators \( \pi \) and \( \rho \).

First we notice that

(P2):

\[
x u(x) = (\pi+\rho) u(x)
\]

that leads to give a general expression of a polynomial \( P(x) \) with respect to \( \pi \) and \( \rho \).

We have: