INTRODUCTION.

Let $K$ be a field of characteristic zero. Consider a homogeneous ideal $I$ in $R = K[x_0, \ldots, x_n]$ generated by homogeneous elements $f_1, \ldots, f_k$ of degree less than $d$, and the following problems arising in various areas:


Call $(R/I)_s$ the homogeneous part of degree $s$ of the graded ring $R/I$; for $s$ large enough, $\dim_K(R/I)_s$ becomes a polynomial in $s$ (the so-called Hilbert polynomial $P(s)$). Define:

$$H(I) = \inf\{s_0 \in \mathbb{N} \mid \dim_K(R/I)_s = P(s) \text{ for } s > s_0\}.$$

What is the smallest upper bound $H(n,d)$ of the $H(I)$'s for all such ideals?

2. Standard bases of $I$.

Choose an order $<$ on the monomials, compatible with products; call $D_{<,x}(I)$ the largest degree of elements of a minimal standard basis of $I$ (relative to $<$). What is the smallest upper bound $D(n,d)$ of the $D(I)$'s for all such ideals?


Consider the $R$-module of relations among the generators $f_1, \ldots, f_k$, i.e., the kernel of $\phi: R^k \rightarrow R$, where $\phi(g_1, \ldots, g_k) = \sum_{i=1}^k f_i g_i$. As usual, $R^k$ is graduated so that $\phi$ is homogeneous of degree 0. What is the smallest integer such that this module can be generated by elements of degree less that it?

More generally, following the notations of B. Angéniol [A] we can introduce the integers $S_i(I)$ $(0 < i < n+1)$. Let $L'$ be a $R$-free resolution of $A$, each $L^i$ being graduated so that $L^i + L^{i-1}$ is homogeneous of degree 0; choose a basis $e_i$ of $L^i$; define $S_i(L', e) = \sup_{h \in e_i} (\deg h) - i + 1$, and $S_i(I) = \inf_{L', e} (S_i(L', e))$. Finally $S(I) = \sup_i S_i(I)$. 


What is the smallest upper bound $S(n,d)$ for all such ideals?

Note that among these integers attached to $I$, $D_{\text{lex}}(I)$ is not intrinsic, and depends not only on the ordering but also on the choice of coordinates. We shall denote it simply by $D(I)$ if we deal with:

- the lexicographic ordering

$$(a_0,\ldots,a_n) <_{\text{lex}} (b_0,\ldots,b_n) \iff \exists j \ (0 \leq j \leq n) \text{ such that } a_0 = b_0,\ldots,a_{j-1} = b_{j-1} \text{ and } a_j < b_j.$$

The exponent of a polynomial $f = \sum_{a \in \mathbb{N}^{n+1}} f_a x^a$ will be $\exp f = \inf_{\text{lex}} \{ a \in \mathbb{N}^{n+1} \mid f_a \neq 0 \}$.

- generic coordinates

following the notations of Galligo ([GA 1]), for $M$ in $GL(n+1)$ denote by $f_M^M$ the polynomial defined by $f_M^M(x) = f(Mx)$, and by $I_M^M$ the ideal generated by $f_1^M,\ldots,f_k^M$. As usual we shall say that a property is true for $I$ in generic coordinates if there exists a non-empty Zariski-open subset of $GL(n+1)$ such that for all $M$ in it, the property holds for $I_M^M$.

All these integers are strongly related, as shown by:

**Theorem A**: $-1 + \frac{S(n,d)}{2^{n-1}} < S_2(n,d) \leq D(n,d) = H(n,d) \leq S(n,d) + n$.

The knowledge of this common upper bound (say $D(n,d)$) is essential for these effectivity problems; computing it exactly seems to be a huge task. Nevertheless we have the following asymptotic behaviour:

**Theorem B**: $D(n,d)$ is bounded by a polynomial in $d$, with leading term $6^{2n-3} d^{3.2n-3}$ (for $n \geq 3$).

On the other hand a family of examples of Mayr-Meyer [M-M] shows that for an infinite number of values of $n$ this double exponentially growth of $D(n,d)$ cannot be avoided. So it seems natural to introduce the real $a$

$$a = \inf \{ a \mid D(n,d) \leq O(d^a) \quad n \to \infty \}$$

Refinements lead to the bounds

$$2^{1/10} = 1.07 \ldots < a < 1.72 \ldots = \sqrt{3} \ .$$

The same number characterizes the asymptotic behaviour of $S_i(n,d)$ for arbitrary $i$, and of $S(n,d)$.

In last part we consider analogous problems for non necessarily homogeneous ideals, proving weaker affine versions of Lazard's results [L 2]; this leads to a pleasant generic linear bound for $H(I)$, $D(I)$ and $S(I)$, contrasting with the double exponent-