CHARACTERIZATION OF CONTROLLABILITY VIA A REGULAR FUNCTION : EXAMPLE OF THE VIBRATING STRING

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Abstract

We study, in the case of a point-controlled vibrating string, two real valued functions of a structural parameter which reflect, in some way, the degree of controllability of the system. These functions may be seen as realistic connections between non-robust binary notions of controllability and the continuous solutions of well-posed control problems. The classical controllability property may be characterized via a regularity argument: the system is controllable if and only if the considered functions are differentiable and non-controllability corresponds to cusp points with negative concavity.

Introduction

Control theory for dynamic systems is closely associated to basic controllability notions. In the infinite dimensional case, there exist various definitions [4], more or less similar to the finite dimensional one, and corresponding to different concepts for the reachable set. Unfortunately, they are often non-robust in the sense that they may no longer be verified for some arbitrarily small variation of system parameters. Consider, for example, the string equation in the domain $]0,1[$, with an internal point actuator: it is classical that such a system is controllable [9] if and only if the support of the actuator is an irrational number. From a practical viewpoint, such an assertion is obviously not sufficient. Indeed, the dependance of any fundamental property of a system should preferably be continuous with respect to structural parameters [1], at least almost everywhere.

Here, the lack of continuity is clearly due to the fact that controllability is a binary concept (together with the infinite-dimensionality of the state space). And although rational position of the actuator does lead to non-controllability, such a distinction is not realistic in practice because small variations of the position parameter cannot be significant.

On the other hand, the quantitative concept of degree of controllability has been early introduced by Kalman, Ho and Nerendra [6], associated to the question: how controllable is the system? In the finite-dimensional case, many papers [5], [7], [12], [14], [15] developed various definitions based on the determinant of the controllability matrix. They possess the following common properties:

- They are continuous with respect to the system parameters,
- They vanish when the system is not controllable,
- They are independent of the initial state of the system.

However, if one try to generalize these definitions to the infinite dimensional case (via convenient adaptations), the aforementioned particularities generate major drawbacks:
the continuity is generally lost due to the non robustness of the controllability notions.

the area of the unit sphere (of initial states) is infinite and it becomes difficult to realize integration on such domain (unless considering a convenient weight measure).

In the case of the point controlled vibrating string (generalization to many other systems are of course possible), we define and study two degrees of controllability, derived from particular real valued functions of a structural parameter. The two difficulties previously mentioned are solved as follow:

- the final state control problem considered in the classical definitions of controllability and degree of controllability, which is generally ill-posed in the distributed case [10], is replaced by two well-posed control problems. The corresponding degrees of controllability depend on an additional parameter which reflects in practice the price one should pay for controlling the system (the more one accepts to pay, the more the system will be controllable).

- In an infinite dimensional state space, balls are not compact; they physically constitute "too large" sets. So, we must consider that the supposed random initial state is constrained by the finite variance additional hypothesis. Physically, such an hypothesis means that the high order modes of the system are of decreasing (and summable) mean energy; this seems us to be a very realistic and little restrictive condition.

The first definition (degree of active controllability) is associated to the classical control problem \( P_e \) : \( \min_u \{ \|X(T)\|^2 + \epsilon \int_0^T u^2(t)dt \} \), seen as a regularization of the exact control problem \( P_0 \) : \( \min_u \{ \int_0^T u^2(t)dt, X(T) = 0 \} \). We show that the considered function is in fact continuous, almost everywhere differentiable and exhibits cusp points when the system is not controllable. So, it may be interpreted like a characterization of binary controllability, via smoothness arguments. Furthermore, it also gives quantitative informations in terms of efficiency of the control. Its computation is essentially based on the knowledge of the eigen-modes of the system, which is of course rather restrictive in practice.

The second definition (degree of passive controllability) is associated to a passive control problem corresponding to a static dissipative feedback (analogous to a viscous damper). The computation is only based on simple numerical simulations, which permits very easy adaptations to various systems. These two approaches, at first sight rather different, lead to very close results. In the case of a boundary control, one could easily show that they are intimately correlated. Indeed, the two considered control problems become equivalent: the transparent boundary condition (which in the physical field corresponds to the well-known notion of matched impedance) is obtained by a static boundary feedback; the induced control \( u \) is in fact the solution of the exact optimal control problem \( P_0 \) (with \( T = 2 \)). This perfect analogy then suggests the question, which was one of the origins of this paper: is it possible, in some sense, to adapt such a property to internal controls? this question, not yet completely solved, has a partial (positive) response in the present analysis.

1 Preliminaries and problem statement

Consider the point-controlled string equation on \( ]0,1[ \times ]0,T[ \) :