NONLINEAR WAVE PROPAGATION
THROUGH A RANDOM MEDIUM
AND SOLITON TUNNELING

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Abstract

We have studied the propagation of non-linear waves across a random medium, using the nonlinear Schrödinger equation with a random potential as a model. By simulating a scattering experiment, we show that non-linearity leads to an improvement of the transmission only when it contributes to create pulses. The propagation properties of these pulses can be described by an equivalent particle theory. Numerical experiments show that this is an approximation: two situations are presented depending on the two ratios of the amplitude and velocity to the perturbation. Concluding remarks link the time-dependent and time-independent regimes.

I. Introduction

We have chosen the nonlinear Schrödinger equation

\[ i \frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + 2 \beta |A|^2 A = \sigma V(x) A \]  

(1)

as a model for the study of the influence of non-linearity on wave propagation across a random medium because it has the simplest possible non-linearity and also because of its canonical character in the description of weakly non-linear effects.

In the linear regime ($\beta = 0$) time and space become decoupled, one can write $A(x, t) = \varphi(x) e^{-i\omega t}$ so that (1) reduces to

\[ \varphi_{xx} + (\omega - \sigma V(x)) \varphi = 0 \]  

(2)

On a discretized version of (2) it can be shown that $\varphi$ grows exponentially with $x$. In terms of a scattering experiment on the slab of random medium of width $L$ where a left incoming wave $A e^{ikz}$ is partially reflected and partially transmitted giving rise to $R(L) e^{-ikz}$ to the left of the medium and $T(L) e^{ikz}$ to the right, it has been shown by Anderson and Ishii [1,2] that $T(L)$ behaves like $T(L) = e^{-L/\delta}$ ($\delta$ is
the localisation length). Souillard and Rammal [3,4] have studied the non-linear situation assuming the ansatz giving rise to equation (3) is still valid. Because of bi-stability effects, the fixed input problem as described previously is now different from the fixed input problem in which $T$ is imposed and not the incoming amplitude. For the physically interesting situation (fixed input) Rammal [4] finds that for small $L (= A^2/\delta)$, $R_1(L) < |R(L)| < R_2(L)$ with large oscillations. $R_1(L)$ and $R_2(L)$ behave like $\sqrt{L}$. For larger $L (> A^2/\delta)$, one recovers the exponential behavior.

For equation (1) these results are questionable because it is not clear how the time-independent non-linear solution evolves into the harmonic form given above. It is also unclear what is the mechanism for the increase of transport in the non-linear regime. To tackle these questions, we have performed the scattering experiment in the full time-dependent framework.

II. The scattering experiment: why we study soliton propagation

To integrate numerically equation (1) with boundary conditions $A$ fixed at $x = \pm \infty$, we have used the following discrete approximation:

$$idA_n/dt + A_{n+1} - 2A_n + A_{n-1} + \beta |A_n|^2(A_{n+1} + A_{n-1}) = \sigma V_n A_n$$

(3)

integrating the equations with a fourth order Adams-Moulton predictor corrector scheme. The particular form of the non-linear term was chosen so that the system is completely integrable when $V_n = 0$ everywhere and the boundary conditions are $q = 0$ at $x = \pm \infty$ [5]. This enables us to test the scheme very accurately and to have stable solutions when the wave-length is not very much bigger than 1.

For the scattering experiment, we write the solution $A_n(t)$ of (3) as the sum of a left incoming wave and an additional term $U$ which we solve for,

$$A_n(t) = A \Delta_n e^{ikn-i\omega(k)t} + U_n(t).$$

The amplitude coefficient $\Delta_n$ is 1. at the left of the slab and decreases to 0. at the boundary of it. The reflected and transmitted waves are absorbed by introducing a damping term in (3). The potential $V_n$ is obtained by averaging a gaussian process $V_0_p$ with an Orstein-Ulhenbeck factor:

$$U_n = \sum_{p=-\infty}^{n} V_0_p e^{-\gamma(n-p)}$$

$\gamma^{-1}$ is the correlation length. Exactly as in [3,4] the non-linearity parameter $\beta$ and $V_n$ have same support.

We first have tested the scheme without randomness. $U_n$ develops into the plane wave when $\beta = 0$, when $\beta > 0$ it decomposes into lumps (Benjamin-Feir instability). These can be related to multi-soliton solutions of the non-linear Schrödinger equation. When $\beta < 0$ we expect a superposition on non-linear plane-wave-like solutions but have not been able to characterise it.