Linearized Maps for the Davey-Stewartson I Equation

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Abstract

Simple equations are derived which express the deformation of the scattering data as a function of the deformation of the potential and vice versa. Applications include the characterization of the higher flows, and computation of the Poisson bracket of scattering data.

1 Introduction

We consider the system of equations

\[ i q_1 + \frac{1}{2} (q_{1xx} + q_{1yy}) - (\varphi_x - q_1 q_2) q_1 = 0, \]
\[ -i q_2 + \frac{1}{2} (q_{2xx} + q_{2yy}) - (\varphi_x - q_1 q_2) q_2 = 0, \]
\[ \varphi_{xx} - \varphi_{yy} - 2 (q_1 q_2)_x = 0. \]

The reduction \[ q_2 = \pm q_1^*, \] leads to the Davey-Stewartson (DS) I system of equations [1]. This system is the shallow water limit of the Benney-Roskes equation [2], where \( q \) is the amplitude of a surface wave packet and \( \varphi \) characterizes the mean motion generated by this surface wave. One assumes small amplitude, nearly monochromatic, nearly one-dimensional waves with dominant surface tension. The DS equation provides a two dimensional generalization of the celebrated nonlinear Schrödinger equation and furthermore can be derived from rather general asymptotic considerations [3].

Introducing characteristic coordinates \( \xi = x + y, \eta = x - y, \) defining

\[ U_1 = -\varphi_\eta + \frac{1}{2} q_1 q_2, \quad U_2 = -\varphi_\xi + \frac{1}{2} q_1 q_2, \]

and integrating equation (1.1.c), equations (1.1) reduce to

\[ \frac{d}{d\xi} U_1 = \frac{1}{2} \frac{d^2}{d\eta^2} \left( \frac{1}{2} q_1 q_2 \right), \quad \frac{d}{d\eta} U_2 = \frac{1}{2} \frac{d^2}{d\xi^2} \left( \frac{1}{2} q_1 q_2 \right). \]

\[ \frac{d^2}{d\xi^2} \left( \frac{1}{2} q_1 q_2 \right) - \frac{d^2}{d\eta^2} \left( \frac{1}{2} q_1 q_2 \right) = 0. \]

\[ i q_{1\xi} + q_{1\eta} + q_{1\eta\eta} + (U_1 + U_2)q_1 = 0, \]
\[ -i q_{2\xi} + q_{2\eta} + q_{2\eta\eta} + (U_1 + U_2)q_2 = 0, \]  
(1.3)
\[ U_1 = -\frac{1}{2} \int_{-\infty}^{\infty} d\xi' (q_1q_2)_{\xi} + u_1, \quad U_2 = -\frac{1}{2} \int_{-\infty}^{\infty} d\eta' (q_1q_2)_{\eta} + u_2, \]
where \( u_1(\eta, t) \neq U_1(-\infty, \eta, t), u_2(\xi, t) \neq U_2(\xi, -\infty, t). \)

An initial value problem associated with equations (1.1), where \( q_1(x, y, 0), q_2(x, y, 0) \) are given and decaying for large \( x, y, \) was solved in [4]. However, recently there has been a renewed interest for equations (1.1): (a) Schultz and Ablowitz [5] have emphasized the importance of the boundary conditions for \( \varphi. \) In particular they have shown that the case solved in [4] corresponds to \( u_1 = u_2 = 0. \)

(b) It has recently been shown [6] that DSI with \( q_2 = -q_1^* \) admits a special type of a localized solution, a two dimensional breather solution, exponentially decaying in both special coordinates. This is a remarkable development since a disappointing feature of all dispersive multidimensional equations studied so far, has been the lack of two dimensional exponentially decaying solitons.

In this note we first review the direct and inverse problem associated with DSI. In §3 we explain the ideas used recently by Santini and the author in order to establish that DSI, for generic nonzero boundary values of \( u_1 \) and \( u_2, \) possesses two dimensional solitons. In §4 we study the linearized maps between scattering data and \( q_1, q_2. \) In §5 we use the results of §4 to: (i) Derive the higher flows in terms of asymptotics of appropriate eigenfunctions. (ii) Show that the evolution of the scattering data is simple if \( u_1 = u_2 = 0. \) This is consistent with the results of §3. (iii) Show how the Poisson brackets of scattering data can be computed.

2 The Direct and Inverse Problem

Equations (1.1) are associated with the Lax equation
\[
(\partial_x + J\partial_y)\Psi + Q\Psi = 0, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}.
\]  
(2.1)

Using characteristic coordinates and letting \( \Psi = \exp[ik(Jx - y)]M, \) the Lax equation for \( M(\xi, \eta, k) \) is solved by the following linear integral equations
\[
M_{11}^+ = 1 - \frac{1}{2} \int_{-\infty}^{\infty} d\xi' q_1 M_{21}^+ \quad M_{12}^+ = -\frac{1}{2} \int_{-\infty}^{\infty} d\xi' q_1 M_{22}^+ e^{i(k - \xi')} \\
M_{21}^+ = \frac{1}{2} \int_{-\infty}^{\infty} d\eta' q_2 M_{11}^+ e^{i(\eta' - \eta)} \quad M_{22}^+ = 1 - \frac{1}{2} \int_{-\infty}^{\infty} d\eta' q_2 M_{12}^+.
\]  
(2.2)

Equations (2.2) are Volterra integral equations, thus \( M^+ \) is analytic in the upper half \( k \)-complex plane. Similarly if \( M^- \) satisfies equations similar to those of (2.2), with the integrals in \( M_{21}^+, M_{12}^+ \) replaced by \( \int_{-\infty}^{\infty} \) and \( \int_{-\infty}^{\infty} \) respectively, it follows that \( M^- \) is analytic in the lower half \( k \)-complex plane.

The eigenfunctions \( M^+, M^- \) are related via the scattering equations
\[
\begin{pmatrix} M_{11}^+(k) \\ M_{21}^+(k) \end{pmatrix} - \begin{pmatrix} M_{11}(k) \\ M_{21}(k) \end{pmatrix} = \int_R d\xi S_{21}(k, \xi) e^{-i\xi\eta - ik\eta} \begin{pmatrix} M_{11}^+(\ell) \\ M_{21}^+(\ell) \end{pmatrix},
\]  
\[
\begin{pmatrix} M_{12}^+(k) \\ M_{22}^+(k) \end{pmatrix} - \begin{pmatrix} M_{12}(k) \\ M_{22}(k) \end{pmatrix} = \int_R d\xi S_{12}(k, \xi) e^{i\xi\eta + ik\eta} \begin{pmatrix} M_{11}^-(\ell) \\ M_{21}^-(\ell) \end{pmatrix},
\]  
(2.3)