Partial differential equations that conserve energy can often be written as infinite dimensional Hamiltonian systems of the following general form

\[ \frac{du}{dt} = JE'(u) \]  

(1.1)

where \( J: X^* \to X \) is a symplectic matrix (i.e., \( JJ^* = -1 \)) and \( E: X \to \mathbb{R} \) is a \( C^2 \) functional defined on some Hilbert space \( X \). Examples of partial differential equations that can be put in this form are the nonlinear Klein-Gordon equation

\[ u_{tt} + i\omega u_t - \Delta u + f(x, |u|^2)u = 0 \]  

(1.2)

and the nonlinear Schrödinger equation

\[ iu_t - \Delta u + f(x, |u|^2)u = 0 \]  

(1.3)

A critical point of equation (1.1) is a point \( \phi \in X \) such that \( E'(\phi) = 0 \). One is interested in the stability of such a critical point and as a first step towards that goal I would like to consider the linearization of equation (1.1) around \( \phi \), i.e.,

\[ \frac{dv}{dt} = JE''(\phi)v + JO(\|v\|^2). \]  

(1.4)

In this paper I will study the spectrum of the operator \( A := JE''(\phi) \), for the following cases.

(1) Nonlinear Schrödinger equation:

\[ \vec{u} = (Reu, Imu) = (u, v); \ X = H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \]

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \ E(\vec{u}) = \frac{1}{2} \int \{|\nabla u|^2 + |\nabla v|^2 + F(x, u^2 + v^2)\} \, dx \]

\[ \frac{\partial F(x, s)}{\partial s} = f(x, s). \]

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Now (1.2) can be written as
\[ \frac{d\tilde{u}}{dt} = JE'(\tilde{u}) \]
Linearizing around a critical point \( \tilde{\phi} = (\phi, 0) \), I have
\[
A = -\Delta + f(x, \phi^2) + 2f'(x, \phi^2)\phi^2 \\
B = -\Delta + f(x, \phi^2) \\
f'(x, \phi) = \frac{\partial f}{\partial \phi}(x, \phi) \\
JE'' = \begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix}
\]

\[ (1.7) \]

(2) Klein-Gordon equation:
\[
\tilde{u} = (Re u, Im u, Re u_t, Im u_t) := (u, z, v, w) \\
X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \\
J = \begin{bmatrix} 0 & J_2 \\ J_2 & 0 \end{bmatrix}; \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
E(\tilde{u}) = \frac{1}{2} \int \left\{ w^2 + z^2 + |\nabla u|^2 + |\nabla u|^2 + F(u^2 + v^2) \right\} dx
\]
Now (1.3) can be written as \( d\tilde{u}/dt = JE'(\tilde{u}) \), notice that equation (1.3) is invariant under the group \( \{ e^{is}: s \in \mathbb{R} \} \), denote by \( \{ T(s): s \in \mathbb{R} \} \) this group in the present setting. Suppose that we are looking for periodic solutions of the form \( T(\omega t)\tilde{\phi}(x) \), then the transformation \( \tilde{u} \to T(\omega t)\tilde{u} \) gives the equation
\[ \frac{d\tilde{u}}{dt} = J(E'(\tilde{u}) - \omega Q'(\tilde{u})) \]  
where \( Q(\tilde{u}) = \frac{1}{2} (J^{-1}T'(0)\tilde{u}, \tilde{u}) = \frac{1}{2} \int (vw - uz)dx \). Hence a periodic solution of (1.3) reduces to a steady-state solution of (1.3a) with the reduced energy \( E(\tilde{u}) - \omega Q(\tilde{u}) \).

Linearizing around a critical point \( \tilde{\phi} = (\phi, \omega \phi, 0, 0) \) I have
\[
JE'' = \begin{bmatrix} 0 & J_2B \\ J_2A & 0 \end{bmatrix} \\
A = \begin{bmatrix} A_0 & -\omega \\ -\omega & 1 \end{bmatrix}, \quad B = \begin{bmatrix} B_0 & \omega \\ \omega & 1 \end{bmatrix} \\
A_0 = -\Delta + f(x, \phi^2) + 2f'(x, \phi^2)\phi^2 \\
B_0 = -\Delta + f(x, \phi^2)
\]

\[ (1.9) \]