Monotonic versus Antimonotonic Exponentiation

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Abstract. We investigate the relationship between the monotonic (→) and the antimonotonic exponentiation (−→) in a type system with subtyping. We present a model in which we can develop both exponentiations at the same time. In this model the monotonic and the antimonotonic exponentiation enjoy a duality, namely \( \alpha \rightarrow \beta = C(\alpha \rightarrow C\beta) \) where \( C \) is the type constructor complement. We give a sound and complete system of axioms for the type system with the type constructors \( \rightarrow, \rightarrow, \cup, \cap, \neg, \top \).

1 Introduction

In a type system with subtyping we have an inclusion relation between types which interacts with the type constructors of the type system according to certain rules. For some type constructors, it is quite clear how such interaction rules have to look, e.g. the product type constructor is monotonic in both arguments, i.e. if \( \alpha, \beta, \gamma \) and \( \delta \) are types, where \( \alpha \) is a subtype of \( \beta \), \( \alpha \subseteq \beta \) and \( \gamma \) is a subtype of \( \delta \), \( \gamma \subseteq \delta \) then we have the following behavior

\[
\alpha \times \gamma \subseteq \beta \times \delta
\]

and everybody will agree, that this is the only reasonable behavior for the product. But for other type constructors, like exponentiation, it is not evident what the adequate behavior is. Our interest lies in studying the interaction between the inclusion relation and the type exponentiation constructor. The exponentiation of two types \( \alpha, \beta \) gives us the function space \( \alpha \rightarrow \beta \). Every function \( f \) of type \( \alpha \rightarrow \beta \), that is \( f : \alpha \rightarrow \beta \), maps objects of type \( \alpha \) to objects of type \( \beta \) and so we have

\[
f : \alpha \rightarrow \beta \Rightarrow \forall x.(x : \alpha \Rightarrow fx : \beta).
\]

This is the minimal requirement for the exponentiation. The relevant literature presents two different constructions which satisfy this requirement.

One is the antimonotonic approach where the exponentiation is antimonotonic in the first and monotonic in the second argument, i.e. if \( \alpha \subseteq \beta \) and \( \gamma \subseteq \delta \) then \( \beta \rightarrow \gamma \subseteq \alpha \rightarrow \delta \). For example, the integers Int are a subtype of the real numbers Real. So, if we have a function \( f : \text{Real} \rightarrow \alpha \) which maps real numbers to objects of type \( \alpha \) then surely \( f \) also maps integers to objects of type \( \alpha \), that is \( f : \text{Int} \rightarrow \alpha \). Therefore we can deduce that \( \text{Real} \rightarrow \alpha \subseteq \text{Int} \rightarrow \alpha \), but not the
converse. We therefore choose as a function space $\alpha \rightarrow \beta$ exactly those functions which maps objects of type $\alpha$ to objects of type $\beta$

$$f : \alpha \rightarrow \beta \iff \forall x. (x : \alpha \Rightarrow fx : \beta).$$

In other words $\alpha \rightarrow \beta$ contains all functions which satisfy the minimal requirement for the exponentiation. Therefore this antimonotonic exponentiation is the greatest possible exponentiation.

The second possibility is the monotonic approach. Here we treat the exponentiation in the same way as the product mentioned above. For the types $\alpha \subseteq \beta$ and $\gamma \subseteq \delta$ we have $\alpha \rightarrow \gamma \subseteq \beta \rightarrow \delta$. The intuition behind this exponentiation is different than in the antimonotonic case. We give the following example to motivate the monotonic approach. We can describe the quality of a wind by the wind direction and the wind force. Knowledge about the wind direction is knowledge of type $D$ and knowledge about both the wind direction and the wind force is knowledge of type $DF$. Hence, $D$ is a subtype of $DF$. If we have a function $f : D \rightarrow \alpha$ which maps knowledge of type $D$ to knowledge of type $\alpha$, then $f$ also maps knowledge of type $DF$ to knowledge of type $\alpha$, that is $f : DF \rightarrow \alpha$ is given simply by ignoring additional knowledge that may be given in $DF$. Therefore we obtain $D \rightarrow \alpha \subseteq DF \rightarrow \alpha$, but not the other way round. So the function space $\alpha \rightarrow \beta$ consists of those functions $f$ which map knowledge of type $\alpha$ to knowledge of type $\beta$ and ignore all knowledge outside of $\alpha$. We obtain

$$f : \alpha \rightarrow \beta \iff \forall x. f(x) = f(x \cap \alpha) : \beta$$

where $x \cap \alpha$ is $x$ restricted to $\alpha$.

Now we can ask whether or not there are other possibilities for the exponentiation, e.g. one that is antimonotonic in the second argument (rather than in the first). But this is impractical, as the following example shows. Let $\bot$ denote the least type and for a type $\alpha$ we have that $\bot \rightarrow \alpha$ denotes the constant functions of $\alpha$. Now, if $\alpha, \beta$ are two types with $\alpha \subseteq \beta$ then we would have the type of constant functions $\bot \rightarrow \beta$ as a subtype of $\bot \rightarrow \alpha$, that is, $\bot \rightarrow \beta \subseteq \bot \rightarrow \alpha$. But normally one has a 1-1 correspondence between the constant functions $\bot \rightarrow \beta$ and the objects of type $\beta$ and we obtain $\beta \equiv \bot \rightarrow \beta \subseteq \bot \rightarrow \alpha \equiv \alpha$.

There is a third possibility; an exponentiation which is neither monotonic nor antimonotonic, i.e. if $\alpha \subseteq \beta$ and $\gamma \subseteq \delta$ then $\alpha \rightarrow \gamma \subseteq \beta \rightarrow \delta$ and $\beta \rightarrow \gamma \subseteq \alpha \rightarrow \delta$. We take as function space $\alpha \rightarrow \beta$ those functions with the property

$$f : \alpha \rightarrow \beta \iff \forall x. (x : \alpha \iff fx : \beta).$$

It is left to the reader to show that this is indeed a valid exponentiation which is neither monotonic nor antimonotonic.

We construct exponentiations with behaviors other than the antimonotonic one by taking appropriate subsets of the function space of the antimonotonic exponentiation. For example, if we define the exponentiation of two types $\alpha, \beta$ as follows

$$f : \alpha \rightarrow \beta \iff fx : \bot \lor (x : \alpha \land fx : \beta)$$