1. THE PROBLEM AND BASIC RESULTS

As in [2] and [3] the following abstract process of vibrations is investigated: Let \( y : [0,T] \rightarrow H \), for any \( T > 0 \), be a function that describes the deviation of a vibrating medium from the position of rest as a function of time \( t \) with values in a (finite- or infinite-dimensional) Hilbert space \( H \). We assume \( y \) to satisfy an abstract wave equation of the form

\[
\ddot{y}(t) - Ay(t) = f(t), \quad t \in (0,T),
\]

where \( \dot{\cdot} \) denotes the derivative with respect to \( t \), \( A \) is a self adjoint positive definite linear operator defined on a dense domain \( D(A) \) in \( H \) and \( f(t) \in H \) for almost all \( t \in [0,T] \), \( \|f(\cdot)\|_H \) is measurable and satisfies

\[
\int_0^T \|f(t)\|^2_H \, dt < \infty \text{ where } \|\cdot\|_H \text{ denotes the norm in } H. \text{ The space of all (classes of) such functions is called } L^2([0,T], H). \text{ Let } N = \dim H. \text{ In addition to the above requirements we assume that } A \text{ has a complete sequence } (\varphi_j)_{j=1}^N \text{ of orthonormal eigenelements } \varphi_j \in D(A) \text{ and a corresponding sequence } (\lambda_j)_{j=1}^N \text{ of eigenvalues } \lambda_j \text{ of finite multiplicity with } 0 < \lambda_1 \leq \lambda_2 \leq \ldots \text{ and } \lim_{j \to \infty} \lambda_j = \infty, \text{ if } N = \infty. \]

Then it follows that

\[
D(A) = \{ v \in H \mid \sum_{j=1}^N \lambda_j^2 \langle v, \varphi_j \rangle_H^2 < \infty \}
\]

and

\[
Av = \sum_{j=1}^N \lambda_j \langle v, \varphi_j \rangle_H \varphi_j \quad \text{for all } v \in D(A).
\]
Furthermore there exists a unique "square root" \( A^{1/2} \) of \( A \) with domain

\[
D(A^{1/2}) = \{ v \in H \mid \sum_{j=1}^{N} \lambda_j <v, \phi_j>^2 < \infty \}
\]

which is defined by

\[
A^{1/2} v = \sum_{j=1}^{N} \lambda_j^{1/2} <v, \phi_j> \phi_j \quad \text{for all } v \in D(A^{1/2}).
\]

Let \( y_0 \in D(A^{1/2}) \) and \( \dot{y}_0 \in H \) be given. Then we require initial conditions for \( t = 0 \) to be given by

\[
y(0) = y_0 \quad \text{and} \quad \dot{y}(0) = \dot{y}_0.
\]

We put \( V = D(A^{1/2}) \) provided with the scalar product

\[
<v, w>_V = \sum_{j=1}^{N} \lambda_j <v, \phi_j>_H <w, \phi_j>_H, \quad v, w \in V.
\]

Then \( V \) becomes a separable Hilbert space which is continuously and densely imbedded into \( H \) because of \( D(A) \subseteq V \subseteq H \) and

\[
||v||_V \geq \lambda_1^{1/2} ||v||_H \quad \text{for all } v \in V.
\]

The dual space \( V^* \) of \( V \) consists of all linear functionals \( v^* : V \to \mathbb{R} \) such that

\[
\sum_{j=1}^{N} \frac{1}{\lambda_j} v^*(\phi_j)^2 < \infty
\]

and

\[
v^*(v) = \sum_{j=1}^{N} <v, \phi_j>_H v^*(\phi_j), \quad v \in V.
\]

If we identify \( H \) with its dual space, we obtain the following chain of continuous and dense imbeddings: \( V \subseteq H \subseteq V^* \).

If we define a linear mapping \( \tilde{A} : V \to V^* \) by

\[
(\tilde{A}v)(w) = <A^{1/2}v, A^{1/2}w>_H = <v, w>_V
\]