Theory of Domains and Nearby *

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Abstract. The author presents a topological approach to the development of the theory of domains.

0. In the present paper we deal with historical, methodological, and mathematical aspects of the theory of domains. A recent book [1] by A. Jung demonstrates a noticeable progress in the theory.

Theory of domains arose in the late 60s in research that was carried out independently by Prof. D. Scott in Oxford and by the author in Novosibirsk. D. Scott was interested in a natural mathematical model for the type-free \( \lambda \)-calculus, whereas the author developed a theory of partial computable functionals of finite types. Both problems were solved quite satisfactorily. The corresponding results were reported by the author at the International Congress of Mathematicians in Nice, 1970 [2] and by D. Scott at the International Congress for Logic, Methodology, and Philosophy of Science in Bucharest, 1971 [3].

It turned out that there was a great resemblance between the mathematical models developed. The exact relation between these models was established in [4]. In particular, the notion of Scott's domain (S-domain) and the one of complete \( f_0 \)-space were proved to be equivalent.

N. Bourbaki in “L'Architecture des mathématiques” distinguishes three basic mathematical structures: algebraic, topological, and that of partial order. All these structures are found in the theory of domains. The approach of D. Scott to the introduction of S-domain by means of (directed-complete) partial orders dominates in the current literature on computer science, though many basic concepts of the theory, e.g., the way-below relation, are rather difficult to comprehend. This was the reason why D. Scott repeatedly returned to the theory of domains, attempting to clarify the foundations. Thus, for this purpose he introduced information systems [5].

In the author opinion, topology should be the basic structure in the development of the theory. The author supposes that the topological approach of [4] is more preferable than the one based on a partial order, both for better reception and for potentially greater generality that is needed if one wants to study domains which contain only constructive points. In the present paper the author will try to substantiate this point of view following the ideas expressed in [4].

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1. Let \((X, T)\) be a topological space (\(T\) is a topology on \(X\), i.e., a family of all open sets). We define a preorder \(\leq_T\) on \(X\), related to the topology \(T\), as follows: for \(x, y \in X\)

\[ x \leq_T y \iff \text{for every open set } V \subseteq X(V \in T)(x \in V \rightarrow y \in V). \]

This relation is a partial order provided that \((X, T)\) is a \(T_0\)-space, i.e., the weakest separation axiom holds: for every \(x, y \in X\), if \(x \neq y\), then there exists an open set \(V \subseteq X\) such that \(x \in V\) and \(y \notin V\), or \(x \notin V\) and \(y \in V\).

The subscript \(T\) in the notation \(\leq_T\) will usually be omitted. We introduce the following notation: \(\hat{x} = \{y \mid y \in X, y \leq x\}, \check{x} = \{y \mid y \in X, x \leq y\}\).

If \((X, T)\) is a \(T_1\)-space (i.e., \(\forall x, y \in X (x \neq y \rightarrow \exists V \in T(x \in V \land y \notin V))\)), then the preorder \(\leq\) degenerates to the identity relation. In the sequel, we will consider only \(T_0\)-spaces.

We introduce one more relation, namely the approximation relation \(\prec\) on elements of \(X\) as follows: for \(x, y \in X\)

\[ x \prec y \iff \text{there exists an open set } V \subseteq X\text{ such that } (y \in V \text{ and } \forall z \in V(x \leq z)). \]

Remark. An equivalent definition may be given as follows: \(x \prec y \iff y \in \text{Int } \hat{x}\), where \(\text{Int } Y\) is the interior of \(Y\), i.e., the largest open subset of the set \(Y \subseteq X\).

Note that \(x \prec y\) implies \(x \leq y\).

We will use the following notation: \(\hat{\hat{x}} = \{y \mid y \in X, y \prec x\}, \check{\check{x}} = \{y \mid y \in X, x \prec y\}\).

We call a topological space \((X, T)\) approximative (or an \(\alpha\)-space) if the following condition holds: for any open set \(V \subseteq X\) and any element \(x \in V\) there exists \(y \in V\) such that \(y \prec x\).

It is easy to see that the following holds:

1. If \((X, T)\) is an \(\alpha\)-space and a set \(V \subseteq X\) is open, then

\[ V = \bigcup_{x \in V} \hat{x}. \]

2. If \((X, T)\) is an \(\alpha\)-space and \(x \in X\), then for every \(y, z\) such that \(y \prec x\) and \(z \prec x\) there exists \(u \prec x\) such that \(y \prec u, z \prec u\).

3. \(x = \sup \hat{x}\), i.e., \(x\) is the least upper bound (relative to the order \(\leq\)) of the set \(\hat{x}\).

Let \((X, T)\) be an \(\alpha\)-space. A set \(X_0 \subseteq X\) is called a base subset of \(X\) if the following condition holds: for any open set \(V \subseteq X\) and any \(x \in V\) there exists \(y \in V \cap X_0\) such that \(y \prec x\).

Remark. \(X\) is a base subset of \(X\).

Remark. If \(X_0\) is a base subset of \(X\), then \(V = \bigcup_{x \in V \cap X_0} \hat{x}\) for every open set \(V \subseteq X\).