NONSMOOTH SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATION

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Introduction

We are concerned here with the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t} + H(t\ ,x\ ,-\frac{\partial V}{\partial x}) = 0\ ,\quad V(1\ ,\cdot) = g \tag{1}$$

arising in optimal control.

It is well known that in general it does not admit classical (C^1) solutions, even when the data are smooth. This led to several weaker notions of solutions (see for instance [5]-[9], [12], [17] and bibliographies contained therein). We do not have the ambition here to provide the reader with a complete overview of existing definitions of solutions, but we shall compare few of them having their origins in Nonsmooth Analysis.

The importance of HJB (Hamilton-Jacobi-Bellman) equation for the investigation of properties of dynamical systems and, in particular, of control systems was recognized a long time ago.

One may see that the value function of an optimal control problem verifies (1) whenever it is smooth. This allows in some (very restrictive) cases to obtain a short proof of Pontriagin's necessary conditions for optimality, to prove some sufficient conditions for optimality and to construct optimal feedbacks [11].

Let us emphasize that the value function arising in control theory is nondecreasing along all trajectories of the system and is constant along optimal trajectories. This leads to a verification technique in optimal control (see [19] for a complete discussion and references). However, "computing" the value function from its definition is a very difficult task.

On the other hand if we are able to find a solution of (1) having the cornerstone properties of the value function, then we may hope to use it for the same purposes.

One can seek for instance to define the solution of HJB equation in such a way that the value function is the unique solution to (1). The notion of viscosity solution introduced in [8], [9], [17] (see also [10] for bibliographical comments) fulfills that objective, but only partially: the uniqueness results are proved up to now only for continuous solutions on open sets. Although a large class of free end point optimal control problems have a locally Lipschitz value function, it is well known that for the target problem and, more generally, for problems with state constraints,

the value function is not continuous and very strong controllability conditions are needed to prove its continuity. Controllability conditions exclude however from consideration a large number of control problems.

A different approach was developed in [5]-[7], where a solution of HJB equation associated to the target problem:

$$\text{minimize } g(x(1))$$

$$\dot{x} = f(t, x, u(t)),\ u(t) \in U \text{ is measurable} \tag{2}$$

$$x(0) = x_0,\ x(1) \in K$$

was defined in such a way that it is locally Lipschitz and nondecreasing along trajectories of the control system (2). Naturally this led to apply the verification technique to functions which may be different from the value function. Such a generalized solution is not uniquely defined, but it allows to consider a broader class of control problems, to adapt the verification technique to problems with discontinuous value function and to get some necessary and sufficient conditions for optimality ([6], [7]).

Another important property of smooth value function is the possibility to construct optimal feedback laws. When the value function is the unique locally Lipschitz viscosity solution of (1) it allows as well to associate feedback laws to the solution of HJB equation. This feature fails whenever the value function is discontinuous. Generalized solutions of (1) defined in [6] enjoy more regularity than the value function does and for the very same reason are only far relatives of this last one:

It is impossible to associate the optimal feedback law with an arbitrary generalized solution of (1). A counter example was constructed in [4].

One would wonder which way to choose. Clearly, we cannot expect from the solution of HJB equation to be unique, locally Lipschitz and at the same time to be equal to the value function.

In this paper we show that a necessary and sufficient condition for a function $V : [0, 1] \times \mathbf{R}^n \longrightarrow \mathbf{R} \bigcup \{\pm\infty\}$ to be nondecreasing along all trajectories of the control system (2) is:

$$\sup_{u \in U} D_{\uparrow}(-V)(t, x)(1, f(t, x, u)) \leq 0 \tag{3}$$

where $D_{\uparrow}(-V)(t, x)$ is the contingent epiderivative of the function $-V$ at (t, x) (see Section 2 for precise definitions).

Such a necessary and sufficient condition leads to a verification technique. Then we investigate necessary and sufficient conditions for a function $V : [0, 1] \times \mathbf{R}^n \longrightarrow \mathbf{R} \bigcup \{\pm\infty\}$ to have the following property: