LOCAL CONTROLLABILITY OF GENERALIZED QUANTUM MECHANICAL SYSTEMS

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ABSTRACT 

The concept of local controllability is investigated for non-relativistic quantum systems. Sufficient conditions will be sought such that the solution of the controlled Schrödinger equation can be guided, over a short time interval, to any chosen point in a suitably prescribed neighborhood of the solution in the absence of control. Evolution equations which are linear in the controls but nonlinear in the quantum state \( \psi \) are considered. Our formulation and analysis will (for the most part) run parallel to those of Hermes. 

I. INTRODUCTION 

In recent years, there has been a growing interest in the system theoretic problems of filtering and control of quantum mechanical systems. Several noteworthy efforts exist: (i) Tarn, Huang and Clark [1] and van der Schaft [2] have explored the formal basis for the modelling of quantum mechanical control systems. (ii) Clark, Tarn and their associates [3–6] have obtained results on quantum nondemolition filtering problem. (iii) Belavkin [7] has investigated the measurement and control problem in quantum dynamical systems. (iv) Pierce, Dahleh and Rabitz [8] have studied the optimal control problem of quantum mechanical systems. (v) Butkovskiy and collaborators have discussed the control of quantum objects in broad terms and have set forth general conditions for controllability of pure quantum states [9–11]. 

To the authors' knowledge very little has been published in the way of mathematically definitive results on the controllability of quantum systems. In [12] the authors are able to establish a series of global controllability conditions for the Schrödinger equation which is linear in state and linear in the external controls by extending the geometric approach as implemented by Sussmann and Jurdjevic [13,14], Krener [15], Brockett [16], Kunita [17] and others.
In the present contribution, we shall consider evolution equations which are linear in the controls but nonlinear in the quantum state; in this case the work of Hermes [18] is extended to obtain conditions for local controllability along an unguided reference solution.

II. PROBLEM FORMULATION WITH NONLINEAR GENERATORS

In adapting Hermes' work [18] to our ends, it is convenient to think in terms of the $x$ representation [19]. Thus the state vector $\xi \in \mathbb{H}$ will be represented by the wave function $\xi(x) \in L^2(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$ stands (ordinarily) for the set of spatial coordinate variables associated with the quantum system. (More generally, $x$ may stand for any complete set of compatible variables [19] built from the position and momentum variables. Spin and other internal degrees of freedom can be incorporated by essentially trivial modifications.) Now, let us define a class of operators $H$ in $\mathbb{H}$ which are supposed to be skew-Hermitian (norm preserving) and time independent and have, in the $x$ representation, the mode of action

$$ (H\xi)(x) = H\xi|_x = \sum_{\lambda=1}^{p} \sum_{\mu=1}^{q} f_{\lambda,\mu}((H_{\lambda,\mu}\xi)(x)) \cdot \xi'(x). $$

Here, $p, q$ are some integers, the $H_{\lambda,\mu}$ ($\lambda = 1,...,p; \mu = 1,...,q$) are closed, skew-Hermitian linear operators acting in $H$, and the mappings $f_{\lambda,\mu}: C^1 \rightarrow C^1$ are real analytic. (By the last requirement we mean that $f_{\lambda,\mu}(w)$ is a real analytic function of its argument $w$, this argument in itself being generally complex, $w \in C^1$. Also, in expression (1), $f_{\lambda,\mu}(w)f_{\lambda',\mu'}(w')$ is to be interpreted as the usual product of complex functions.) Throughout the current section, the generators $H_0,\ldots,H_x$ entering the "controlled Schrödinger equation" will be assumed to be of this more general form. Thus, while $H_0,\ldots,H_x$ are still taken skew-Hermitian, they need not be linear—although the linear case is certainly included.

We shall further assume that a unique local solution exists for the initial value problem

$$ \frac{d}{dt} \psi_t = \left[ H_0 + \sum_{j=1}^{r} u_j(t)H_j \right] \psi_t, \quad \psi_{t=0} = \phi \in \mathbb{H}, $$

posed by the Schrödinger equation so generalized, the admissible controls $u_j$ now being real, analytic, bounded functions of $t$. To establish that this is a viable assumption, we note that it is automatically fulfilled within the framework of [12], provided $\phi$ belongs to the analytic domain $D_\phi$; moreover, in Ref. 20 it has been shown to be valid for a certain relevant class of partial differential equations.

On the other hand the formulation of general conditions on $H_0 + \sum u_jH_j$ for the