VIABILITY TUBES

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Abstract

We define viability tubes and invariant tubes of a differential inclusion, we study some asymptotic properties and we characterize them by showing that the indicator functions of their graphs are solutions to the contingent Hamilton-Jacobi equation. We provide some examples of viability tubes.

Contents

Introduction

Let X be a finite dimensional vector space and $F : [o\ \infty[\times X \to X$ a set-valued map which associates with any state $x \in X$ and any time t the subset $F(t\ ,x)$ of velocities of the system. The evolution of the system is governed by the differential inclusion

(*) $x'(t) \in F(t\ ,x(t))\ ,x(t_0) = x_0$

We consider now "tubes", i.e., set-valued maps $t \to \underline{P}(t)$ from $[o\ ,\infty[$ to X. We say that a trajectory $t \to x(t) \in X$ is "viable" (in the tube $\underline{P}$) if

(**) $\forall t \geq 0\ ,x(t) \in \underline{P}(t)$

A tube $\underline{P}$ enjoys the viability property if and only if, for all $t_0 \geq 0$ and $x_0 \in \underline{P}(t_0)$, there exists at least a solution $x(\cdot)$ to the differential inclusion (*) which is viable.

Remark

A simple-valued tube $t \to \{x(t)\}$ enjoys the viability property if and only if $x(\cdot)$ is a solution to the differential inclusion (*). So it is legitime to regard a tube having the viability property as a "multivalued solution" to the differential inclusion (1).

The knowledge of a tube enjoying the viability property allows to infer some informations upon the asymptotic behaviour of some solutions to the differential inclusion (1), as we do with Lijapunov functions. They also share the same disadvantages: the dynamics F being given, how do we construct the tubes of F?

We shall begin by characterizing such tubes as "viability tubes". For that purpose, we need an adequate concept of derivative of set-valued map, the "contingent derivative" defined as follows:

If $x \in \underline{P}(t)$, v belongs to $D\underline{P}(t\ ,x)$ (1) if $\liminf_{h \to 0+} d(v\ ,\frac{\underline{P}(t+h)-x}{h}) = 0$

Viability tubes are those tubes satisfying

(***) $\forall t \geq 0\ ,\forall x \in \underline{P}(t)\ ,F(t\ ,x) \cap D\underline{P}(t\ ,x)(1) \not\subset \phi$

We can regard (***) has a "differential equation for tubes".

We prove in the second section that the "limit" when $t \to \infty$ of a viability tube $\underline{P}(t)$ (namely, the Kuratowski limsup) is a viability domain: hence targets of a differential inclusion are necessarily viability domains. We construct in the fourth section the largest viability tube "converging" to a given target. We also provide a surjectivity criterion which is useful for solving such problems.

We can characterize viability tubes $\underline{P}(t)$ by the indicator functions $V_{\underline{P}}$ of their graphs, defined by: $V_{\underline{P}}(t\ ,x) := 0$ if $x \in \underline{P}(t)$, $+\infty$ if not. We thus observe that $\underline{P}$ is a viability tube if and only if $V_{\underline{P}}$ is a solution to the "contingent Hamilton-Jacobi equation".

$$\inf_{v \in F(t\ ,x)} D_+ V(t\ ,x)(1\ ,v) = 0$$

where

$$D_+ V(t\ ,x)(1\ ,v) := \liminf_{\substack{h \to 0+ \\ v' \to v}} \frac{V(t+h\ ,x+hv') - V(t\ ,x)}{h}$$

is the contingent epiderivative of V at $(t\ ,x)$ in the direction $(1\ ,v)$.

We then investigate tubes enjoying a dual property, the ***invariance property***: for all $t_0 \geq o$ and $x_0 \in \underline{P}(t_0)$, *all* solutions to the differential inclusion are viable.