STABILITY OF SEMILINEAR SYSTEMS IN
HILBERT SPACES

PIOTR GRABOWSKI
Academy of Mining and Metallurgy
30-059 Kraków,Mickiewicza 30,Poland

1. Abstract semilinear systems.

Several problems of mathematical physics, control and circuit theories lead to the following semilinear problem

\[ \dot{x}(t) = Ax(t) + B F[x(t)] \] (1.1),

where \( x(t) \in X \) for every fixed \( t \geq 0, X \) is a real Hilbert space with scalar product \( \langle \cdot , \cdot \rangle_X \); \( A : (2(\mathbb{H}) \subset X) \rightarrow X \) is a linear operator, which is the generator of a linear \( C_0 \)-semigroup on \( X \); \( U \) is another real Hilbert space with scalar product \( \langle \cdot , \cdot \rangle_U \); \( B \in L(U,X) \).

It follows from the earlier results due to Segal [1],azy [2] and Ball [3] that for every initial condition \( x_0 \in X \) there exists a unique weak (mild) solution of (1.1) prolongable on its right maximal interval of existence \( [0,t_{\text{max}}(x_0)] \), provided that \( P:X \rightarrow U \) is a locally Lipschitz function.

The aim of this paper is to give sufficient conditions for global (as well as global uniform) asymptotic stability of the equilibrium \( 0 \in X \), independently of \( P \) from the prescribed subclass of locally Lipschitz functions from \( X \) into \( U \), vanishing at 0.

2. Main results.

The main results will be formulated as theorems and remarks.

Theorem 1.

Let \( \mathcal{M} \) be a class of functions such that:

(1) \( \mathcal{M} \subseteq \{ P: P \text{ is a locally Lipschitz mapping from } X \text{ into } U, P(0) = 0 \} \),

(ii) There exist operators \( Q \in L(U,X), p=^{\text{unique}} E L(U), \hat{L} \in L(U,X), E=^{\text{unique}} E L(U) \)

such that:
\( \forall F \in \mathcal{M} : QF \) is a gradient type operator,
\( \forall x \in X, \forall F \in \mathcal{M} : \)
\[
\left\langle x, -\Delta x \right\rangle_X + \left\langle x, L^F(x) \right\rangle_X + \left\langle F(x), L^x \right\rangle_U + \left\langle F(x), -K^F(x) \right\rangle_U \geq 0 \tag{2.1},
\]
\( \left( H_3 \right) \) \( \exists F \in \mathcal{M} \in L(X) \) and a real positive \( \varepsilon \) such that
\[
\left\langle Ax, v \right\rangle_X + \left\langle x, H^* Ax \right\rangle_X + \left\langle x, \Delta x \right\rangle_X + \left\langle x, H^* Bu \right\rangle_X + \frac{1}{2} \left\langle u, A^* Ax \right\rangle_U + \frac{1}{2} \left\langle u, B^* Bu \right\rangle_U + (2.2),
\]
\( \left( H_4 \right) \) If \( F \in \mathcal{M} = L(X, U) \cap \mathcal{M} \) then \( A + BF \) generates an exponential-stable semigroup,
\( \left( H_5 \right) \) \( \forall F \in \mathcal{M}, \forall x \in X, x \neq 0 \) \( \exists \alpha = \alpha(x, F) \in \mathcal{M} \)
\[
\left\langle \int_0^1 \left\langle x, \frac{d}{ds} F(sx) \right\rangle_X \right\rangle_X ds = 0,
\]
Then the equilibrium is globally asymptotically stable (GAS) for every \( F \in \mathcal{M} \).

**Sketch of the proof.**

Let us consider a continuously Fréchet-differentiable functional
\[
V(x) = \left\langle x, Hx \right\rangle_X + \int_0^x dy, F(y) \right\rangle_X - \epsilon \left[ \left\| x \right\|^2_X + \left\| F(x) \right\|^2_U \right] \quad \forall x \in \mathcal{D}(\lambda), \forall F \in \mathcal{M} \tag{2.3}\]

The second term in (2.3) is the antiderivative of the gradient operator \( QF \). The assumptions \( \left( H_1 \right), \left( H_2 \right), \left( H_3 \right) \) allow us to prove the following inequalities:
\[
\left\langle Ax + BF(x), V'(x) \right\rangle_X \leq -\epsilon \left[ \left\| x \right\|^2_X + \left\| F(x) \right\|^2_U \right] \quad \forall x \in \mathcal{D}(\lambda), \forall F \in \mathcal{M} \tag{2.4}
\]

where \( V'(x) \) denotes the gradient of \( V \) at \( x \), and
\[
\mathcal{V}[x(t, x_0)] - V(x) \leq -\varepsilon \int_0^t \left\{ \left\| x(t) \right\|^2_X + \left\| F[x(t)] \right\|^2_U \right\} dt \quad \forall x_0 \in X,
\]
\( \forall t \in [0, t_{\text{max}}], \forall F \in \mathcal{M} \)

If \( V \) is nonnegative on \( X \) for every \( F \in \mathcal{M} \) then we can prove that \( t_{\text{max}} = +\infty \) and
\[
\varepsilon \int_0^\infty \left\{ \left\| x(t) \right\|^2_X + \left\| F[x(t)] \right\|^2_U \right\} dt \leq V(x_0) \quad \forall x_0 \in X, \forall F \in \mathcal{M} \tag{2.6}
\]

The conditions assuring that \( V(x) \geq 0 \quad \forall x \in X, \forall F \in \mathcal{M} \) can be derived using the linear comparison system technique. The linear comparison