In this note we present a scheme for estimation of parameters that can be considered an extension of the techniques and ideas of [5], [9] to allow treatment of identification problems that are typical in the 1-D seismic inverse problem [1], [10], [11]. It is shown in [5] how one can use cubic spline approximation techniques in parameter identification problems for hyperbolic systems with simple Dirichlet or Neumann boundary conditions. Here we are again interested in hyperbolic systems but with special boundary conditions which depend on unknown parameters. One possible approach is to make a change of the variables so as to reduce the problem to one with simple known boundary conditions where the unknown parameters have been transformed to the partial differential equation itself. While such a technique can prove fruitful for certain classes of problems (e.g., see the beam examples with damping in [3]), it is not feasible for the problems under consideration here. Rather we shall treat the boundary conditions and unknown parameters contained therein directly.

The problem we consider concerns the acoustic or 1-D elastic wave equation [1], [10], [11] with elastic boundary conditions at one (the upper or left) boundary and absorbing boundary conditions at the other (lower or right) boundary. Specifically we consider

\[
\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( E(x) \frac{\partial u}{\partial x} \right) \quad 0 \leq x \leq 1, \quad t > 0. \tag{1.1}
\]
\[
\frac{3u}{3x}(t,0) + q_1u(t,0) = s(t,\tilde{q}), \quad \frac{3u}{3t}(t,1) + q_2\frac{3u}{3x}(t,1) = 0, \quad (\ldots 1)
\]
\[
u(0,x) = u_t(0,x) = 0,
\]
where \( q_1 \) is a parameter (an elastic modulus) for the restoring force in the medium at the surface \((x = 0)\), \( s \) is an unknown source term (which we do not assume is necessarily an impulse) resulting from a perturbing shock to the medium at the surface. Here \( q_2 = \sqrt{E(1)/\rho(1)} \) in the absorbing boundary condition (no upgoing or reflected waves) at the "bottom" of the field results from factoring the wave operator at \( x = 1 \), \( \rho \) is the mass density of the medium, and \( E \) is an elastic modulus.

The fundamental problem consists of estimating \( \rho, E, q_1, q_2, \tilde{q} \) from observations of displacement \( u(t,0) \) (or velocity \( u_t(t,0) \)) at the surface. There is a large literature on 1-D seismic inverse problems of this nature and it is well-known that it is, in general, impossible to determine both field mass density and elastic modulus from surface observations alone. It is therefore standard practice to make some assumptions in order to simplify the problem and reduce ill-posedness. Thus the problem we discuss \((\rho = \text{constant})\) is a restriction to a special case of the actual 1-D seismic problem of interest. However, we hasten to add that all the 1-D problems themselves fall short of addressing the "real" problems which are unquestionably 3-dimensional in nature.

Our purpose here is to indicate that methods developed and used in other contexts ([2],[3],[4],[5],[7],[8],[9]) are, in principle, applicable to seismic problems. Even though we employ a simple 1-D model problem to illustrate the ideas, the techniques are readily applicable to higher dimensions and indeed we have already established that certain aspects and features of our schemes can be adapted with relative ease to treat 2-D and 3-D problems.

We observe that in (1), the assumption \( \rho = \text{constant} \) leads to a problem in which knowledge of \( E, q_1 \) and \( q_2 = \sqrt{E(1)/\rho} \) along with that of the source parameters \( \tilde{q} \) resolves the inverse problem. This is the problem for which we have developed both theory and software packages based on the cubic spline approximation techniques of [5]. For ease in exposition here, we discuss the special case were \( E \) is constant and also assume that we have transformed the system (by a standard change of variables) to one with homogeneous boundary conditions. Thus the problem we discuss is the following.

Consider the system
\[
\begin{align*}
    u_{tt} &= q_0u_{xx} + f(t,x,q) \quad 0 \leq x \leq 1, \; t > 0 \\
    u_x(t,0) + q_1u(t,0) &= 0 \quad (2\ldots)
\end{align*}
\]