BOUNDARY FEEDBACK STABILIZATION PROBLEMS FOR
HYPERBOLIC EQUATIONS *)

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In line with the Workshop's expressed request, we intend to present here a brief
summary of recent results in the area of boundary feedback stabilization for
hyperbolic equations, along with some new results of more recent origin. We shall
mainly consider three stabilization problems and draw our material mostly from our
papers [2] - [5].

Let $\Omega$ be a bounded open domain in $\mathbb{R}^\nu$ with boundary $\Gamma$ assumed to be on $(\nu-1)$-
dimensional variety with $\Omega$ locally on one side of $\Gamma$. For the most part, in particular
for the second and third problem considered below, $\Gamma$ may have finitely many conical
points with $\Omega$ convex. In the first two problems, the feedback is an "interior
observation" of the "position", which acts in the Dirichlet B.C., while in the third
problem the feedback is a "boundary observation" of the velocity, which acts in the
Neumann B.C.

Problem 1 (Almost periodic stabilization; no damping) [2]
We begin with a differential operator $A(\xi, \partial)$ which, along with homogeneous Dirichlet
B.C., is realized as an operator $A: L^2(\Omega) \supset D(A) \rightarrow L^2(\Omega)$.
It is assumed throughout that $A$ generates a strongly continuous (s.c) cosine operator
$C(t)$ on $L^2(\Omega)$. We then consider the hyperbolic equation

$$\begin{align*}
x_{tt}(t, \xi) &= -A(\xi, \partial)x(t, \xi) \quad t > 0, \xi \in \Omega \\
x(0, \xi) &= x_0(\xi), \quad x_t(0, \xi) = x_1(\xi) \quad \xi \in \Omega \\
x(t, \xi) &= f(t, \xi) \quad t > 0, \xi \in \Gamma.
\end{align*}$$

(P1.1)

Here, we study the case where $f(t, \xi)$ is realized as a bounded, finite rank operator,
acting only on the position $x$ in the interior (no damping) of the form

$$f(t, \xi) = \sum_{j=1}^{J} <x(t, \cdot), w_j(\cdot)> g_j(\xi)$$

(P1.2)

*) Paper presented at the Workshop by the second named author.
where here and here after \( \langle \cdot, \cdot \rangle \) denotes the \( L^2(\Omega) \)-inner product. We first need examine the well-posedness of the "closed loop system" (P1.1) - (P1.2).

**Theorem 1.1.** [2],[3]. Let \( w_j \in D(cI-A)^{\gamma}P_i \), \( \gamma > 0 \) for some \( c \) for which the fractional powers are well defined. Then, the feedback closed loop solutions \( x(t,x_0,x_1) \) of (P1.1) - (P1.2) can be expressed simply as

\[
x(t,x_0,x_1) = C_F(t)x_0 + S_F(t)x_1, \quad x_0 \in L^2(\Omega), \quad x_1 \in H^{-1}(\Omega), \quad t \in \mathbb{R}
\]

where \( C_F(t) \) defines a s.c. (feedback) cosine operator on \( L^2(\Omega) \) and \( S_F(t) \) is the corresponding sine operator. Actually, \( C_F \) extends/restricts as a s.c. cosine operator on each fixed interpolation space between \( [D(cI-A)^{3/4}P_i] ' \) and \( D(A^{\gamma}P_i) \).

With the wellposedness question settled, we now turn to a problem which may be viewed as being part of the general area of stabilization. We then assume that the operator \( A \) be selfadjoint and unstable, in the sense that its eigenvalues \( \{-\lambda_k\} \) satisfy

\[
\ldots < -\lambda_k < 0 < -\lambda_{k-1} < \ldots < -\lambda_2 < -\lambda_1 \quad \text{(P1.3)}
\]

and are all simple (multiplicity one). Let \( \{\phi_k\} \) denote the corresponding orthonormal basis of eigenvectors in \( L^2(\Omega) \). Thus, the free system \( \{f(t,z) = 0\} \) has the eigensolutions for \( 1 \leq k \leq K-1 \) that blow up exponentially in time. We then pose the problem: can we select general classes of vectors \( w_j \in L^2(\Omega) \), \( g_j \in L^2(\Omega) \) for \( j = 1,2,\ldots, \) minimum which will restore the typical oscillatory behavior of all solutions of the closed loop system (P1.1) - (P1.2)? An answer in spectral terms of the feedback generator \( A_F \) corresponding to \( C_F(t) \) of Theorem 1.1 is given by

**Theorem 1.2.** [2]. Let \( \nu = \dim \Omega > 2 \) and let \( \Omega \) either have \( C^\infty \)-boundary \( \Gamma \) or else be a parallelepiped. Let \( A \) be selfadjoint with simple eigenvalues satisfying (P1.3). Let the vectors \( w_j \in L^2(\Omega) \) satisfy the following algebraic conditions at the unstable eigenvalues

\[
\text{rank } W = l_w \quad \text{with} \quad K-1 \leq l_T + l_w - 1
\]

where \( l_T \) is the number of linearly independent Neumann traces \( \{\partial \phi_k/\partial n\}_r \), \( k = 1,\ldots,K-1 \), and \( W = [w_1,w_2,\ldots,w_{K-1}] \) with

\[
W_k = [\langle w_1,\phi_k \rangle, \langle w_2,\phi_k \rangle, \ldots, \langle w_j,\phi_k \rangle].
\]

Finally, let the vectors \( w_j \) satisfy the growth condition: