1. Introduction

We first give a non-stochastic prototype of the problem under consideration here. Suppose that $M$ is a surface imbedded in three-dimensional space $\mathbb{R}^3$ such that every geodesic disc of radius $\varepsilon$ has area $\pi \varepsilon^2$; then $M$ is flat, in the (intrinsic) sense that every small disc can be mapped isometrically to a flat disc in the plane $\mathbb{R}^2$. Indeed, the area of every small disc determines the Gaussian curvature which in turn gives the required isometry.

In trying to generalize this result to higher dimensional manifolds, geometers have encountered difficulties--beginning with the hypothesis that the volume of every geodesic ball of radius $\varepsilon$ is the same as the volume of the ball of radius $\varepsilon$ in Euclidean space $\mathbb{R}^n$. Therefore it is natural to replace the volume by other measurements.

In this report we give results on the first and second moments of the exit time of Brownian motion, obtaining more effective characterizations of Euclidean and rank one symmetric spaces. Much of the work is in collaboration with Alfred Gray [4], to whom go thanks for many useful conversations.

2. Statement of Results

Let $M$ be an $n$-dimensional Riemannian manifold with Laplace-Beltrami operator $\Delta$. The Brownian motion $(X_t, P_x)$ is a diffusion process with infinitesimal generator $\Delta$. This can be constructed, for example, by solving a stochastic differential equation on the orthonormal frame bundle of $M$ [5]. The generator and the process are connected by Dynkin's identity [3], expressed as follows.

$$f(x) - E_X f(X_T) = - E_X \int_0^T \Delta f(X_s) ds$$

where $f$ is a smooth function on $M$ and $T$ is a stopping time with $E_X(T)$ finite.

Let $T_\varepsilon$ be the exit time from the geodesic ball of radius $\varepsilon$ centered at $m \in M$:

$$T_\varepsilon = \inf \{ t > 0 : d(X_t, m) = \varepsilon \}$$

and let the normalized moments be defined by

Supported by Air Force Office of Scientific Research AFOSR 80-0252A.
$u_0 = 1$

$$u_k(x) = E_x (T_e^k)/k! \quad (k = 1, 2, \ldots)$$

$u_k$ coincides with the classical solution of $\Delta u_k = -u_{k-1}$ with the boundary condition that $u_k = 0$ on the boundary, $k = 1, 2, \ldots$. Therefore, computation of these moments is reduced to solving these partial differential equations. In case $M$ is a Euclidean or rank one symmetric space these equations have radial solutions $u = u(r), r = d(x, m)$ which can be used to give closed formulas for the moments of the exit time [4].

On a general Riemannian manifold a closed form solution is not available. In the following section we develop a perturbation theory for the Laplacian on small geodesic balls. This is used to prove the following asymptotic expansions.

**Theorem 2.1** We have for small $\varepsilon > 0$

$$E_m(T_e) = c_0 \varepsilon^2 + c_1 \varepsilon^4 + \varepsilon^6 [c_2 |R|^2 + c_3 |\rho|^2 + c_4 \tau^2 + c_5 (\Delta \tau)] + 0(\varepsilon^8)$$

$$E_m(T_e^2) = d_0 \varepsilon^4 + d_1 \varepsilon^6 + \varepsilon^8 [d_2 |R|^2 + d_3 |\rho|^2 + d_4 \tau^2 + d_5 (\Delta \tau)] + 0(\varepsilon^{10})$$

Here $c_0, \ldots, d_5$ are constants which depend only on $n = \dim M; R, \rho, \tau$ are respectively the curvature tensor, the Ricci tensor and the scalar curvature, computed at $m \in M$. (cf. Section 4)

Using the exact values of the constants, we have

**Corollary 2.2** Suppose that $M$ is a Riemannian manifold such that for every $m \in M$ and every $\varepsilon > 0$, $E_m(T_e) = \varepsilon^2/2n$. Then $M$ is flat provided that any of the following hold

(i) $n = \dim M \leq 5$

(ii) $M$ is Einstein, or more generally

(iii) $M$ has non-negative (or non-positive) Ricci curvature

One may ask if the restrictions imposed by Corollary 2.2 might be removed if one could compute the coefficient of $\varepsilon^8$ in the expansion of $E_m(T_e)$. The following example shows that this program is fruitless.

**Example 2.3** Let $M = G \times G^c$ where $G$ is a compact semi-simple Lie group and $G^c$ denotes the non-compact dual of $G$. Then $E_m(T_e) = \varepsilon^2/2n + O(\varepsilon^{10})$

Finally, we have a positive result which is valid in all dimensions. Let $V_m(T_e) = E_m(T_e^2) - E_m(T_e)^2$ be the variance of the exit time from the geodesic ball of radius $\varepsilon$.

**Corollary 2.4** Suppose that $M$ is a Riemannian manifold such that for every $m \in M$ and every $\varepsilon > 0$, 