Tracking Property: a Viability Approach

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Abstract

This paper is devoted to the characterization of the tracking property connecting solutions to two differential inclusions or control systems through an observation map derived from the viability theorem. The tracking property holds true if and only if the dynamics of the two systems and the contingent derivative of the observation map satisfy a generalized oartial differential equation, called the *contingent differential inclusion.* This contingent differential inclusion is then used in several ways. For instance, knowing the dynamics of the two systems, construct the observation map or, knowing the dynamics of one system and the observation map, derive dynamics of the other system (trackers) which are solutions to the contingent differential inclusion.

It is also shown that the tracking problem provides a natural framework to treat issues such as the zero dynamics, decentralization, and hierarchical decomposition.

Introduction

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and the *system of differential inclusions*

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

We further introduce a set-valued map $H : X \rightsquigarrow Y$, regarded as an *observation map.*

We devote this paper to many issues related to the following *tracking property*: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exist solutions $(x(\cdot), y(\cdot))$ to the system of differential inclusions such that

$$\forall\, t \geq 0, \;\; y(t) \in H(x(t))$$

The answer to this question is a solution to a *viability problem*, since we actually look for a solution $(x(\cdot), y(\cdot))$ which remains viable in the graph of the observation map H. So, if the set-valued maps F and G are Peano[1] maps and if the graph of H is closed, the Viability Theorem states that the tracking property is equivalent to the fact that the graph of H is a viability domain of $(x, y) \rightsquigarrow F(x, y) \times G(x, y)$.

Recalling that the graph of the contingent derivative $DH(x, y)$ of H at a point (x, y) of its graph is the contingent cone[2] to the graph of H at (x, y), the tracking property is then equivalent to the *contingent differential inclusion*

$$\forall\, (x, y) \in \mathrm{Graph}(H), \;\; G(x, y) \cap DH(x, y)(F(x, y)) \neq \emptyset$$

[1] A set-valued map is called *Peano* if its graph is nonempty and closed, its values are convex and its growth linear.

[2] The contingent cone $T_K(x)$ to a subset K at $x \in K$ is the closed cone of directions $v \in X$ such that $\lim_{h \to 0+} d_K(x + hv)/h = 0$. It is equal to X when x belongs to the interior of K, coincides with the tangent space when K is smooth and to the tangent cone of convex analysis when K is convex. We say that K if *sleek* at x is $y \rightsquigarrow T_K(y)$ is lower semicontinuous at x. In this case, the contingent cone $T_K(x)$ is convex. Convex subsets are sleek.

If (x, y) belongs to the graph of a set-valued map $H : X \rightsquigarrow Y$, the *contingent derivative* $DH(x, y)$ of H at (x, y) is the set-valued map from X to Y defined by

$$\mathrm{Graph}(DH(x, y)) := T_{\mathrm{Graph}(H)}(x, y)$$

We observe that when F and G are single-valued maps f and g and H is a differentiable single-valued map h, the contingent differential inclusion boils down to the more familiar ***system of first-order partial differential equations***[3]

$$\forall\, j = 1, \ldots, m, \ \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x, h(x)) - g_j(x, h(x)) \ = \ 0$$

Since the contingent differential inclusion links the three data F, G and H, we can use it in three different ways:

1. — Knowing F and H, find G or selections g of G such that the tracking property holds (observation problem)

2. — Knowing G (regarded as an *exosystem*, following Byrnes-Isidori's terminology) and H, find F or selections of f of F such that the tracking property holds (tracking problem)

3. — Knowing F and G, find observation maps H satisfying the tracking property, i.e., solve the above contingent differential inclusion.

Furthermore, we can address other questions such as:

a) — Find the largest solution to the contingent differential inclusion (which then, contains all the other ones if any)

b) — Find single-valued solutions h to the contingent differential inclusion which then becomes

$$\forall\, x \in K, \ \ 0 \in Dh(x)(F(x, h(x))) - G(x, h(x))$$

In this case, the tracking property states that there exists a solution to the *"reduced" differential inclusion*

$$x'(t) \ \in \ F(x(t), h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows to divide the system by half, so to speak.

The observation and the tracking problems are the two sides of the same coin because the set-valued map H and its inverse play the same roles whenever we regard a single-valued map as a set-valued map characterized by its graph.

Consider then the observation problem: the idea is to observe solutions of a system $x' \in F(x, y)$ by a system $y' \in G(x, y)$ where $G : Y \rightsquigarrow Y$ describes simpler dynamics: equilibria, uniform movement, exponential growth, periodic solutions, etc. This would allow to observe complex systems[4] $x' \in F(x)$ in high dimensional spaces X by simpler systems $y' \in G(y)$ or even better, $y' = g(y)$, in low dimension spaces. We can think of H as an observation map, made of a small number of ***sensors*** taking into account uncertainty or lack of precision.

For instance, when $G \equiv 0$, we obtain constant observations. Observation maps H such that $F(x) \cap DH(x, y)^{-1}(0) \neq \emptyset$ for all $y \in H(x)$ provide solutions satisfying

$$\forall\, t \geq 0, \ \ x(t) \in H^{-1}(y_0) \ \text{ where } \ y_0 \in H(x_0)$$

In other words, inverse images $H^{-1}(y_0)$ are closed viability domains[5] of F. *Viewed through such an observation map, the system appears in equilibrium.*

[3]For special types of systems of differential equations, the graph of such a map h (satisfying additional properties) is called a ***center manifold***. Theorems providing the existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory.

[4]We can use this tracking property as a ***mathematical metaphor*** to model the concept of .... metaphors in epistemology. The simpler system (the model) $y' \in G(y)$ is designed to provide ***explanations*** of the evolution of the unknown system $x' \in F(x)$ and the tracking property means that the ***metaphor*** H is valid (***non falsifiable***). Evolution of knowledge amounts to "increase" the observation space Y and to ***modify*** the system G (replace the model) and/or the observation map H (obtain more experimental data), checking that the tracking property (the validity or the consistency of the metaphor) is maintained.

[5]When $Y := \mathbf{R}$, such maps can be called "prime integrals" (or "energy functions") of F, because when both $F := f$ and $H := h$ are single-valued, we find the usual condition $h'(x) \cdot f(x) = 0$.