ON INVERSE PROBLEMS FOR EVOLUTIONARY SYSTEMS: GUARANTEED ESTIMATES AND REGULARIZED SOLUTIONS

A.B. Kurzhanski

International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria

I.F. Sivergina

Institute of Mathematics and Mechanics of the Ural Scientific Center
Academy of Sciences of the USSR, Sverdlovsk, USSR

This paper deals with the selection of an initial distribution in the first boundary-value problem for the heat equation in a given domain $[0,\theta] \times \Omega$, $\theta < \infty$ with zero values on its boundary S so that the deviation of the respective solution from a given distribution would not exceed a preassigned value $\gamma > 0$. The result is formulated here in terms of the "theory of guaranteed estimation" for noninvertible evolutionary systems. It also allows an interpretation in terms of regularization methods for ill-posed inverse problems and in particular, in terms of the quasiinvertibility techniques of J.-L. Lions and R. Lattes.

1. The Problem.

Assume Ω to be a compact domain in $\mathbf{R}^n$ with a smooth boundary S; $\theta > 0$, $\gamma > 0$ to be given numbers, functions $y(t,x)$, $z(x)(\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^1)$, $(\mathbf{R}^n \to \mathbf{R}^1)$ to be given and such that $y(\cdot,\cdot) \in L_2([0,\theta] \times \Omega)$, $z(\cdot) \in L_2(\Omega)$.

Denote $u = u(t,x;\, w(\cdot))$ to be the solution to the boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0\,, \quad 0 \le t \le \theta\,, \tag{1}$$

$$u|_{[0,\theta]\times S} = 0\,,$$

$$u|_{t=0} = w(\cdot)\,,$$

Also denote

$$J(w(\cdot)) = \alpha \int_0^\theta \int_\Omega (u(t,x;\, w(\cdot)) - y(t,x))^2 dx dt + \tag{2}$$

$$+ \beta \int_{\Omega} (u(\theta,x;\, w(\cdot)) - z(x))^2 dx$$

with $\alpha \geq 0$, $\beta \geq 0$.

Consider the following problem: among the possible initial distributions $w(\cdot) \in L_2(\Omega)$ specify a distribution $w^0(\cdot)$ that ensures

$$J(w^0(\cdot)) \leq \gamma\,. \tag{3}$$

The latter is an *inverse problem* [1]. With $\alpha = 0$ it was studied by J.-L. Lions and R. Lattes within the framework of the method of "quasiinvertibility" [2]. Numerical stability was ensured in this approach.

Let us now transform the previous problem into the following: among the distributions $w(\cdot) \in L_2(\Omega)$ determine *the set $W^*(\cdot) = \{w^*(\cdot)\}$ of all those distributions $w^*(\cdot)$ that yield the inequality*

$$J(w^*(\cdot)) \leq \gamma\,.$$

Assuming that the problem is solvable ($W^*(\cdot) \neq \phi$) we may describe its solution in terms of the theory of "guaranteed observation" [3]. Namely, assume $y(t,x)$, $z(x)$ to be the available measurements of the process (1), so that

$$y(t,x) = u(t,x;\, w(\cdot)) + \xi(t,x) \tag{4}$$

$$z(x) = u(\theta,x;\, w(\cdot)) + \sigma(x)$$

$$0 \leq t \leq \theta, \ \ x \in \Omega$$

where $\xi(t,x)$, $\sigma(x)$ stand for the *measurement noise* which is *unknown* in advance *but bounded* by the restriction

$$\alpha \int_0^{\theta} \int_{\Omega} \xi^2(t,x)\, dx dt + \beta \int_{\Omega} \sigma^2(x) dx \leq \gamma\,. \tag{5}$$

Then $W^*(\cdot)$ will be precisely the *set of all initial states* of system (1) *consistent with measurements $y(t,x)$, $z(x)$* (4) *and with restriction* (5).

The aim of this paper will be to describe some stable schemes of calculating the sets $W^*(\cdot)$ and their specific elements. (A direct calculation of these may obviously lead to unstable numerical procedures.)