A Linear Time Algorithm to Recognize Clustered Planar Graphs and Its Parallelization

Elias Dahlhaus

Department of Mathematics and Department of Computer Science,
University of Cologne
dahlhaus@informatik.uni-koeln.de
and
Dept. of Computer Science
University of Bonn
dahlhaus@cs.uni-bonn.de
Germany

Abstract. We develop a linear time algorithm for the following problem: Given a graph $G$ and a hierarchical clustering of the vertices such that all clusters induce connected subgraphs, determine whether $G$ may be embedded into the plane such that no cluster has a hole. This is an improvement to the $O(n^2)$-algorithm of Q.W. Feng et al. [6] and the algorithm of Lengauer [12] that operates in linear time on a replacement system. The size of the input of Lengauer's algorithm is not necessarily linear with respect to the number of vertices.

Introduction

In VLSI-design and in drawing figures, the following problem comes up. The nodes of a graph are partitioned into subdivisions (clusters) and and the clusters are again divided into clusters and so on. One would like to embed the nodes in a cluster quite closely. In VLSI-design one is interested to put nodes into the same cluster that belong to the same electronic unit (see for example [8]). Other applications appear in software visualization [19] and in knowledge representation [9].

The ideal case would be when the graph could be embedded into the plane, such that edges do not cross and the clusters look "nicely". That means clusters should appear as connected areas without holes.

One algorithm that recognizes clustered graphs with connected clusters is due to C.W. Feng [6]. The clusters are given by a cluster tree. In her structure, the size of the input is in the order of the number of vertices. The time bound is $O(n^2)$. A very first algorithm is due to Lengauer [12]. The input is given by a replacement system. The size of the input might exceed the order of the number of vertices. The time bound is linear in the order of the input size, but not linear in the order of the number of vertices.

As in [6], we assume that the clusters induce connected subgraphs of the given graph. This is quite reasonable, because vertices in the same cluster should be close to each other with respect to adjacency.
Here we present an $O(n)$ time algorithm to recognize clustered planar graphs. The algorithm is based on the decomposition of a graph into its "3-connected components". The major idea of the new algorithm is that we weight the clusters by their sizes and weight each edge by the weight of the cluster of minimum size that contains it. We also weight each face by the maximum weight of its adjacent edges. We shall see that a planar embedding is a clustered planar embedding if and only if for each number $i$, the faces of weight $\geq i$ together with the edges of weight $\geq i$ form a connected subgraph of the dual graph.

In Sect. 2, we introduce the notation that is necessary for the whole paper. In Sect. 4, we discuss the decomposition of planar graphs into "uniquely" embeddable components. In Sect. 3, we show the key result that characterizes clustered planar embeddings as mentioned in the last paragraph. In Sect. 5, we introduce the recognition algorithm for clustered planar graphs, show its correctness, and analyze its sequential and parallel complexity. In the last section, we finish with some concluding remarks.

2 Notation and Basic Definitions

A graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$. Multiple edges and loops are not allowed.

An induced subgraph is an edge-preserving subgraph, that means $(V', E')$ is an induced subgraph of $(V, E)$ iff $V' \subset V$ and $E' = \{xy \in E : x, y \in V'\}$.

Trees are always directed to the root. The notion of the parent, child, ancestor, and descendant are defined as usual. We denote by $n$ the number of vertices and edges of $G$. In planar graphs $n$ is in the order of the number of vertices.

We call a graph $k$-connected if for each pair $a$ and $b$ of vertices, there are $k$ from $a$ to $b$ that have pairwise $a$ and $b$ as only common vertices.

A graph is planar if it can be embedded into the plane, such that edges do not cross. An edge crossing free embedding of a planar graph into the plane is also called a planar embedding. The areas the plane is subdivided by a planar embedding are called the faces of the planar embedding. The combinatorial embedding of a planar embedding of $G$ consists of the clockwise enumerations of the incident edges of the vertices of $G$. Two combinatorial embeddings $(f_v)_{v \in V}$ and $(g_v)_{v \in V}$ are equivalent if for each vertex $v$, the enumerations $f_v$ and $g_v$ define the same cyclic orientation of the incident edges of $v$. The reversal of a combinatorial embedding $(f_v)_{v \in V}$ is the combinatorial embedding $(g_v)_{v \in V}$ that comes up by reversing the enumerations of incident vertices. Two combinatorial embeddings are weakly equivalent if they are equivalent or one is equivalent to the reversal of the other. The dual graph $D_{emb}$ of a planar embedding $emb$ of $G$ consists of the set $F$ of faces of $emb$ as vertices the edges of $D_{emb}$ are the pairs of faces that share an edge of $G$. Note that each face of $emb$ are determined by the counterclockwise enumeration of its incident edges and the faces are uniquely determined by the combinatorial embedding associated with $emb$.

Lemma 1. (see for example [11]) All combinatorial embeddings of a 3-connected planar graph are weakly equivalent. □