On the Non-Trivial Concept of Relaxation in N-body Systems

Piero CIPRIANI 1 2 & Giuseppe PUCACCO 2 3

1 Scuola di Dottorato di Ricerca in Fisica,
2 International Center for Relativistic Astrophysics,
   University of Rome “La Sapienza”, I-00185 Roma, Italy
3 II University of Rome “Tor Vergata”, I-00133 Roma, Italy

1. Introduction

There is no doubt that to speak about relaxation towards equilibrium in N-body self-gravitating systems is not an easy task. The difficulties are related, first of all, to the elusivity, in this context, of the very meaning of equilibrium; consequently, one must content oneself with the study of the processes through which the system forgets progressively some memory of the initial conditions. So, one has not to describe the relaxation to equilibrium, but, rather, to trace several histories, the plots of which have different players performing on different time-scales.

We surely know of two of these processes: the first one is the establishing of the dynamical equilibrium; the second (but we could equally well say the last) corresponds to “something similar” to the approach to the canonical equilibrium of statistical mechanics. It is a common opinion [4],[5] that in order to reach the dynamical equilibrium (starting from an arbitrary initial bounded state) the system needs a time of the order of the crossing time ($\tau_C \sim R/\sigma$, where $R$ is a characteristic length scale and $\sigma$ is the RMS velocity), whereas to reach equipartition of energy it needs a time of the order of $N\tau_C/\ln N$ ($\sim \tau_B$, the binary relaxation time-scale). One more point of consensus is that if a system tends to forget its initial conditions on a certain time-scale (say $\tau_S$), it must also display a chaotic behaviour, namely its dynamics must be extremely sensitive to small changes in initial condition. A fundamental question is therefore the following: is $\tau_S$ related to $\tau_C$ or to $\tau_B$ or to some other (eventually intermediate) time-scale? The main lines of thinking on this topic are two: a) $\tau_S \sim \tau_C$ (a result obtained by Kandrup [15],[16],[17] and by Goodman, Heggie & Hut [12], with different approaches and also with a different interpretation as far as the sources of instability); b) $\tau_S \sim N^{1/3}\tau_C$ (Gurzadyan & Savvidy [13]).
The aims of the present contribution are to point out that the relationships between the instability time-scale, as measured from the exponential divergence of trajectories, both from numerical integrations and from "averaged" analytical estimates, the mixing time, if it can be estimated in some way, and the "relaxation" time (when it is possible to speak about one relaxation time) are in general highly non-trivial.

So, a very cautious treatment of these concepts is needed in order to avoid some contradictions and misunderstandings, which, in our opinion are mainly due to some unjustified extrapolations and approximations made in (most of) the treatments presented in the last years.

Clearly, the most self-consistent approach to the problem outlined above should be properly carried out in the framework of general Hamiltonian many-body dynamics, and this has already been done\[21\] for a class of many degrees of freedom Hamiltonians of interest mainly in solid state physics, and working in a slightly different manifold. The work along this line is going\[7],[8\]; nevertheless our concern here is to point out that most of the conclusions drawn until now, concerning in particular the $N$-body self-gravitating problem, should be analysed carefully, and that the contradictions between analytical estimates based on different approximations, and the results found by numerical simulations are due to a set of excessive simplifications of the geometro-dynamical quantities governing the evolution of perturbations, on one side; and also to an inadequate representation of the dynamics, on the other side.

In particular, if we can observe that the semi-analytical estimate of the "collective relaxation time-scale", made by Gurzadyan & Savvidy\[13\], is based on some unrealistic schematizations, nevertheless, the general setting of the problem and the references to the rigorous results of ergodic theory are stated in a nearly correct way.

On the other hand, it is also certainly true that the time-scale of instability (i.e. of "exponential growth") of perturbations in a $N$-body self-gravitating systems is of the order of the crossing time, but, in principle, this is not necessarily related with any kind of relaxation, and in particular, with the mixing property of dynamical systems.

2. Crossing and Collective Relaxation Time-Scales

To test the sensitivity of a system to changes in initial conditions the first simple approach is to compare the evolutions followed by the same system starting from a set of different, but very close, initial conditions: let us consider, for simplicity, two points in the $6N$-dimensional phase space of the system and, choosing in some appropriate way a "distance" in the phase space, let us trace it in time as the two trajectories evolve from the initial points. If the system dynamics is sensitive to initial conditions, this distance