HARMONIC ANALYSIS ON REDUCTIVE \textit{p}-ADIC GROUPS

(after HARISH-CHANDRA [4 (c)])

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In [7], Mautner gives a method for constructing irreducible unitary representations of the \textit{p}-adic group $\text{PGL}(2)$, whose matrix-coefficients are square-integrable functions. For this purpose he starts with an irreducible unitary representation $\tau$ of the open compact subgroup $K$, being the canonical image in $\text{PGL}(2)$ of the group of integer matrices with determinant a unit in $\text{GL}(2)$, whose restriction to the subgroup generated by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ does not contain the identity representation. The unitary representation of $\text{PGL}(2)$ induced by $\tau$, decomposes into a direct sum of finitely many irreducible representations, whose matrix-coefficients with respect to a suitable orthonormal base, are continuous functions with compact support.

These representations are special cases of so-called supercuspidal representations. They are defined as follows.

Let $\mathbb{Q}$ be a \textit{p}-adic field, i.e. a locally compact field with a non-trivial discrete valuation. We start with a connected, reductive (linear) algebraic group $G$ defined over $\mathbb{Q}$ and we denote by $G$ its subgroup of $\mathbb{Q}$-rational points. Then $G$ is a locally compact, separable and unimodular group. Let $P$ be a parabolic subgroup of $G$, defined over $\mathbb{Q}$, with unipotent radical $N$. Then $N$ is defined over $\mathbb{Q}$ as well. We put $P = P \cap G$, $N = N \cap G$ and we call $P$ a parabolic subgroup of $G$ with unipotent radical $N$. $P$ and $N$ determine $P$ and $N$ completely. By $Z$ we denote the maximal $\mathbb{Q}$-split torus in the center of $G$. We write $Z = Z \cap G$. Let $f$ be a continuous function on $G$ with compact support mod $Z$. For any parabolic subgroup $P$ of $G$ with unipotent radical $N$, put

$$F^P(x) = \int_{N} f(xn) dn \quad (x \in G)$$

where $dn$ is a fixed Haar measure on $N$. 
A continuous (complex-valued) function \( f \) on \( G \) is said to be a supercuspidal form if

(i) \( \text{Supp} \ f \) is compact \( \text{mod} \ Z \) ;

(ii) \( f^P = 0 \) for all parabolic subgroups \( P \not\subseteq G \).

Let \( \pi \) be an irreducible unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \). We call \( \pi \) supercuspidal if there exist \( \varphi, \psi \in \mathcal{H} - \{0\} \) such that the function

\[
x \mapsto (\varphi, \pi(x)\psi) \quad (x \in G)
\]

is a supercuspidal form.

In fact, Mautner's method of construction for this class of representations in case \( \text{PGL}(2) \), can be generalized to semisimple \( G \) by taking a "good" open compact subgroup of \( G \) (given by Bruhat-Tits) and a so-called supercuspidal representation of \( K \), i.e. an irreducible unitary representation of \( K \) whose matrix-coefficients, considered as functions on \( G \), are supercuspidal forms. It is an open problem if one obtains all supercuspidal representations of \( G \) by means of the above construction. There is another, more serious, motivation for studying supercuspidal representations. It originates from the "philosophy of cusp forms" which appeared to be very fruitful in harmonic analysis on reductive groups ([2], [4(a)]). Furthermore we have to mention that the supercuspidal representations occur naturally in the study of automorphic forms (cf. [5]).

Of course our main goal here is the Plancherel formula for \( G \). Apart from technical difficulties (no differential operators, no good "exp" in positive characteristic) we are faced with a fundamental problem: given any irreducible unitary representation of \( G \), does its character exist?

For supercuspidal representations the answer is affirmative. In the next paragraphs we propose to describe some of the results concerning supercuspidal representations. Proofs are omitted almost everywhere. They can be looked up in [4(c)].