1. Introduction. An arithmetical function $f(n)$ is called additive if for all coprime $m$ and $n$,

$$f(mn) = f(m) + f(n),$$

and strongly additive if (1) holds and if for all integers $a \geq 1$ and for any prime $q$,

$$f(q^a) = f(q).$$

$f(n)$ is called strongly multiplicative if, in addition to (2), the relation

$$f(mn) = f(m)f(n)$$

holds for $(m,n) = 1$.

Let $u(n)$ be an arithmetical function and put

$$F_N(x) = \sum_{n \leq N, u(n) \leq x} 1.$$  

We say that $u(n)$ has an asymptotic distribution (or limit law), if there is a distribution function $F(x)$ with $F_N(x) \to F(x)$ for all continuity points of the latter. Our aim is to give conditions for an arithmetical function $u(n)$ to have a limit law, with main emphasis on the classes of additive and multiplicative functions. This theory has a wide literature and a good account of works can be found in Kubilius [16] and [18] and Galambos [9] and [11]. The present paper is a continuation of my work in [8] and [9].

The concept of a limit law defined in terms of $F_N(x)$ suggests that our problem is of probabilistic character. As a matter of fact, considering the probability space $S_N = (X_N, G_N, P_N)$ with $X_N = \{1, 2, \ldots, N\}$ and $P_N$ generating the uniform distribution on the set $G_N$ of all subsets of $X_N$, i.e., $P_N([i]) = 1/N$ for all $i$, any arithmetical function $u(n)$ is a random variable on $S_N$ and
hence well known methods of probability theory are applicable to find the limit of $F_N(x)$.

Let $f(n)$ be a real-valued strongly additive function. We have by induction from (1) and (2) that

\[ f(n) = \sum f(q) = \sum_{k \geq 1} f(q_k) e_k(n) \]

where

\[ e_k(n) = \begin{cases} 1 & \text{if } q_k | n, \\ 0 & \text{otherwise}. \end{cases} \]

Here, and in what follows, $q_1 < q_2 < \cdots$ denotes the increasing sequence of prime numbers. By definition

\[ P_N(e_k = 1) = \frac{1}{N/q_k} = 1/q_k + O(1/N) \]

and similarly

\[ P(e_{k_1} = 1, \ldots, e_{k_t} = 1) = 1/q_{k_1} \cdots q_{k_t} + O(1/N). \]

(8) and (9) imply that the random variables $\{e_k\}$ are "almost independent" in the sense of probability theory and (6) thus shows that an additive arithmetical function $f(n)$ is the sum of "almost independent" random variables on $S_N$. Using Fourier transforms, the continuity theorem of characteristic functions (see Loeve [20], p. 192) yields that there is a limit law for $f(n)$ if the limit, as $N \to +\infty$, of

\[ E_N\left[ \exp\left( it \sum f(q_k) e_k(n) \right) \right] = \frac{1}{N} \sum_{n=1}^{N} e^{itf(n)} \]

exists and if it is continuous at $t = 0$. The almost independence of the $e_k$'s will help in approximating the left hand side of (10) by a product and it will then be easy to obtain conditions on $f(n)$ to guarantee the existence of the limit of the left hand side of (10).