5.1. Introduction. In this chapter, we will consider the saturation of positive operators in the space \( C[a,b] \). The situation here differs considerably from the periodic case. For one thing, transform techniques which were so useful in the periodic case are no longer applicable. In fact, even the notion of convolution, which was the primary method of constructing operators in \( C^*[-\pi,\pi] \) does not carry over as nicely to \( C[a,b] \).

To make this point more clear, let's consider a sequence of non-negative continuous functions \( h_n \) and define the operators \( L_n \) on \( C[a,b] \) by

\[
L_n(f,x) = \int_a^b f(t) h_n(t-x)dt.
\]

Consider the simplest function \( f(t) = 1 \) on \([a,b] \). Then

\[
L_n(f,a) = \int_a^b h_n(t-a)dt = \int_0^{b-a} h_n(t)dt,
\]

and

\[
L_n(f,b) = \int_a^b h_n(t-b)dt = \int_0^{a-b} h_n(t)dt.
\]

If \( (L_n) \) is to be an approximating sequence then the functions \( h_n \) must concentrate their mass close to 0 as \( n \to \infty \) and these masses must converge to 1. This forces either \( \int_0^{b-a} h_n(t)dt \) or \( \int_{a-b}^0 h_n(t)dt \) to converge to something \( \leq \frac{1}{2} \) as \( n \to \infty \) and thus

\[
L_n(f,a) \neq f(a) \quad \text{or} \quad L_n(f,b) \neq f(b)
\]
What we have pointed out is that such a sequence of operators cannot be used to approximate each function $f \in C[a,b]$ uniformly on $[a,b]$, because there is always difficulty at the end points of the interval. However, such a sequence is still useful for approximating on intervals $[c,d] \subseteq [a,b]$ if $a < c < d < b$. We have seen at least one important example of this in the Landau operators introduced in 2.5.

Another common way of constructing operators in $C[a,b]$ is by an interpolation formula

$$L(f,x) = \sum_{k=1}^{m} f(x_k) h_k(x)$$

where $a \leq x_1 < x_2 < \ldots < x_m \leq b$ and for each $k$, $h_k$ is a continuous and non-negative function on $[a,b]$. The Bernstein operators, Hermite-Fejér operators, and variation diminishing spline operators (see Chapter 2) are examples of this form.

5.2. Saturation in $C[a,b]$. The setup for the saturation problem in $C[a,b]$ is slightly different from that in $C^*$. Where as in $C^*$, we had that only the constant functions were approximated with precision "o" of the saturation order, we now wish to loosen this requirement by saying only that certain trivial functions are approximated with order "o" of the saturation order. This trivial class will depend on the sequence of operators and will have to be determined as part of the saturation problem. A second change is that we will now allow the saturation order to be a sequence of functions rather than just a sequence of real numbers. We need this in order to accommodate the possibility that the degree of approximation is effected by the position of the point $x$ in the interval. For example, we have seen in Chapter 2 that approximation by Bernstein polynomials is better near the end points of the interval than it is