7.1. Introduction. Saturation is a way of measuring the efficiency of approximation by a sequence of operators \((L_n)\). When \((L_n)\) is saturated with order \((\phi_n)\) (or \((\phi_n(x))\)), then we know that no amount of smoothness on the function \(f\) to be approximated can guarantee an error of approximation \(o(\phi_n)\). However, saturation gives no information as to what functions are approximated with an order \((\psi_n)\), which is not optimal (i.e. \(\psi_n \neq O(\phi_n)\)). There are two common ways of measuring the efficiency of approximation in the non-optimal case: approximation of classes of functions, and inverse theorems. In this chapter, we will discuss problems in the approximation of classes of functions while in Chapter 8 we will deal with inverse theorems.

If \(A\) is a class of functions from \(C^*[\pi, \pi]\) (or \(C[a,b]\)), we define the error in approximating \(A\) by \(L_n\) to be

\[
E(A, L_n) = \sup_{f \in A} ||f - L_n(f)||
\]

where of course the norm is taken in the range of \(L_n\). If \(A\) is compact, then \(E(A, L_n) \to 0\). In the case that, \(L_n(1) = 1\), it is not necessary to have \(A\) compact for \(E(A, L_n) \to 0\).

The problem in approximating classes of functions is to determine \(E(A, L_n)\). For some operators \(L_n\) and classes \(A\), this is possible, but generally this is too difficult. Instead, we will usually have to settle for an asymptotic analysis of \((E(A, L_n))\) as \(n \to \infty\). For us, this analysis will take one of two forms. One is to find a sequence \((\psi_n)\) such that
\[ E(A,L_n) \sim (\psi_n) \quad (7.1.2) \]

(See Section 3.1 for the notation of \( \sim \)). This, we call a weak asymptotic analysis of \( E(A,L_n) \). A second more precise analysis is to find \( (\psi_n) \) such that

\[ E(A,L_n) = \psi_n + o(\psi_n), \quad (n \to \infty) \quad (7.1.3). \]

We will call (7.1.3) a strong asymptotic analysis of \( E(A,L_n) \).

It is clear that this is not the strongest possible asymptotic analysis of \( E(A,L_n) \). However, finer estimates are usually more delicate and have only been obtained for specific examples.

If \((L_n)\) is any sequence of positive operators on \( C[a,b] \) and

\[ \mu_n^2 = \| L_n((t-x)^2, x) \| \]

then for any class \( A \) which contains multiples of \( 1, x \) and \( x^2 \), we know

\[ E(A,L_n) \geq \mu_n^2 \quad (7.1.4). \]

Also, if \( A \) is contained in the class \( W^{(1)}(1,M) = \{ f' \in \text{Lip}(1,M) \} \) then the estimate

\[ E(A,L_n) \leq C \mu_n^2 \quad (7.1.5) \]

follows from our estimates in Chapter 2 (Theorem 2.3). Thus, for example, the error in approximating any of the classes \( A = W^{(r)}(a,M) \) with \( r \geq 2 \) is