CHAPTER 7

APPROXIMATION OF CLASSES OF FUNCTIONS

7.1. Introduction. Saturation is a way of measuring the efficiency of approximation by a sequence of operators \((L_n)\). When \((L_n)\) is saturated with order \((\tilde{\phi}_n)\) (or \((\phi_n(x))\)), then we know that no amount of smoothness on the function \(f\) to be approximated can guarantee an error of approximation \(o(\tilde{\phi}_n)\). However, saturation gives no information as to what functions are approximated with an order \((\psi_n)\), which is not optimal (i.e. \(\psi_n \neq O(\phi_n)\)). There are two common ways of measuring the efficiency of approximation in the non optimal case: approximation of classes of functions, and inverse theorems. In this chapter, we will discuss problems in the approximation of classes of functions while in Chapter 8 we will deal with inverse theorems.

If \(A\) is a class of functions from \(C^k[\pi,\pi]\) (or \(C[a,b]\)), we define the error in approximating \(A\) by \(L_n\) to be

\[
E(A, L_n) = \sup_{f \in A} \|f - L_n(f)\|
\]

(7.1.1)

where of course the norm is taken in the range of \(L_n\). If \(A\) is compact, then \(E(A, L_n) \to 0\). In the case that, \(L_n(1) = 1\), it is not necessary to have \(A\) compact for \(E(A, L_n) \to 0\).

The problem in approximating classes of functions is to determine \(E(A, L_n)\). For some operators \(L_n\) and classes \(A\), this is possible, but generally this is too difficult. Instead, we will usually have to settle for an asymptotic analysis of \((E(A, L_n))\) as \(n \to \infty\). For us, this analysis will take one of two forms. One is to find a sequence \((\psi_n)\) such that
(See Section 3.1 for the notation of \(~\)). This, we call a weak asymptotic analysis of \(E(A, L_n)\). A second more precise analysis is to find \((\Psi_n)\) such that

\[
E(A, L_n) = \Psi_n + o(\Psi_n), \quad (n \to \infty)
\]

We will call (7.1.3) a strong asymptotic analysis of \(E(A, L_n)\).

It is clear that this is not the strongest possible asymptotic analysis of \(E(A, L_n)\). However, finer estimates are usually more delicate and have only been obtained for specific examples.

If \((L_n)\) is any sequence of positive operators on \(C[a, b]\) and

\[
\mu_n^2 = \|L_n((t-x)^2, x)\|
\]

then for any class \(A\) which contains multiples of \(1, x\) and \(x^2\), we know

\[
E(A, L_n) \geq \mu_n^2
\]

Also, if \(A\) is contained in the class \(W^{(1)}(1, M) = \{f' \in \text{Lip}(1, M)\}\) then the estimate

\[
E(A, L_n) \leq C \mu_n^2
\]

follows from our estimates in Chapter 2 (Theorem 2.3). Thus, for example, the error in approximating any of the classes \(A = W^{(r)}(a, M)\) with \(r \geq 2\) is