1. INTRODUCTION

In the steady-state operation of a bare, homogeneous, monoenergetic reactor, the neutron density $u(x)$ satisfies the boundary value problem

$$\Delta u + \omega u = 0, \quad x \in D; \quad u = 0, \quad x \in \partial D,$$

where $D$ is the domain (taken in $\mathbb{R}^n$, for generality) occupied by the reactor, $\partial D$ is its boundary, and $\omega$ is a positive parameter incorporating the absorption cross section, the multiplication factor and the diffusion constant. Normally $\omega$ can be varied by adjusting the position of the control rods within the reactor.

We have previously considered (Payne and Stakgold, [8]) the linear problem where $\omega$ is independent of $u$ and the reactor is operating at criticality. This means that $\omega = \lambda_1$, where $\lambda_1$ is the fundamental (lowest) eigenvalue of

$$\Delta \varphi + \lambda_1 \varphi = 0, \quad x \in D; \quad \varphi = 0, \quad x \in \partial D.$$

It is well known that $\lambda_1$ is simple and positive with an associated eigenfunction $\varphi_1(x)$ which may be chosen positive in the open set $D$. With this agreement, $\varphi_1(x)$ is determined within a positive multiplicative constant so that the form, but not the size, of the neutron density is specified. This underlines the severe limitation of any linear theory, namely, that many quantities of physical importance such as the operating power of the reactor cannot be calculated within the theory.

Nevertheless the linear model predicts accurately the value of $\omega$ at criticality and the general form of the neutron density. In our paper, we were able to obtain bounds for the gradient of the neutron density in terms of its maximum value and thus develop isoperimetric inequalities for the mean-to-peak neutron density ratio which depends only on the shape of the reactor.

We shall consider here a simple nonlinear version of (1) with $\omega$ a function of the steady temperature, which, when measured with respect to a suitable datum, is itself proportional to the neutron density. We recast (1) in the form

$$-\Delta u = f(u), \quad x \in D; \quad u = 0, \quad x \in \partial D,$$

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where
\[ f(z) = \lambda z - p(z), \quad \lambda > 0, \]
\[ p(z) \in C^2(-\infty, \infty), \quad p(0) = p'(0) = 0, \quad p(z) > 0 \text{ for } z > 0. \]

We have written (3) with \(-\Delta u\) on the left side to stress the analogy with steady heat conduction for which a positive value of the right hand side characterizes a source of heat, whereas a negative value characterizes a sink. With this physical background in mind, questions of existence and notions of upper and lower solutions discussed in section 2 often acquire a straightforward intuitive meaning. Since \(f(0) = 0\), we are dealing with what D. Cohen and H. B. Keller have called the unforced problem. In (4), \(p(z)\) is positive for \(z > 0\) in keeping with the usual physical requirement of negative feedback. The case \(p(z) = z^2\) is particularly important and has been considered by Smets [11] in one dimension (when the solution is an elliptic function), by Kastenberg and Chambre [4], and, in the forced case, by D. Cohen [1] who was concerned with power excursions in a reactor. Outside the specific reactor setting, equation (3) has been widely studied; probably the most general results on positive solutions can be found in H. B. Keller [6] and in D. Cohen and B. Simpson [2].

In section 2, we study the existence and uniqueness of positive solutions for (3) and (4), sharpening slightly some of the results previously known. In section 3, we derive a maximum principle for (3) enabling us to obtain bounds for \(|\text{grad } u|\). We then illustrate the use of these bounds in isoperimetric inequalities and norm estimates.

2. EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS

We are interested in positive solutions \(u(x)\) of (3). Our first results do not require the specific assumptions (4) on \(f(z)\). Instead we shall only demand that \(f(0) = 0\) and that \(f(z) \in C^2(-\infty, \infty)\). For simplicity in formulating the theorems, we shall exclude the linear case \(f(z) = \lambda_1 z\) and the case where \(f(z)\) coincides with \(\lambda_1 z\) in some finite neighborhood of \(z = 0:\)

\[ (\text{NL}) \quad z = 0 \text{ is an isolated zero of the equation } f(z) - \lambda_1 z = 0. \]

Our first theorem is a slight generalization of one by H. B. Keller [5].

**Theorem 1.** If \(f(z)\) satisfies (NL) and either