II. Elimination of extensionality

In Takeuti [60], Gandy [8] and Schütte [47] the axiom of extensionality is eliminated by relativization. Here we develop this idea in a form specially suitable for functional languages. For \( (\omega AC) \), \( (C) \) and some special cases of \( (AC) \) the elimination is carried out. This yields a \( (E1) \)-elimination for \( (AC)^O \)-resp. \( (\omega AC) \)-analysis with the axioms \( (AC)^{\alpha, \beta \text{-qf}} \) restricted to the types \( (\alpha \equiv 0, \beta \text{ arbitrary}), (\alpha \equiv 0 \ldots 0, \beta \equiv 0) \).

We content us here with this result because by chapter I it covers classical analysis. - In the following chapters proof-theoretic reductions are given for \( (AC)^O \)-resp. \( (\omega AC) \)-analysis with extensionality restricted to \( (ER) \)-qf. Thus we investigate \( (AC)^O \), \( (\omega AC) \)-analysis with the above restrictions on \( (AC) \)-qf as well as \( (AC)^O \), \( (\omega AC) \)-analysis with extensionality restricted to \( (ER) \)-qf.

Definition of Ex, \( \alpha \) by induction on types:

I. Ex°(r ) =: \( \forall \), r°e° s ° e: r s

II. Ex (r ) =: \( \forall x_1, \ldots, x_m \forall y_1, \ldots, y_m \exists x_1, \ldots, x_m r x_1 \ldots x_m r y_1 \ldots y_m \)

\( \alpha \in \mathbb{E} \) =: Ex (r ) \( \vee \) Ex (s ) \( \vee \exists x_1, \ldots, x_m \exists x_1, \ldots, x_m r x_1 \ldots x_m r y_1 \ldots y_m \)

for \( \alpha \equiv \alpha_1 \ldots \alpha_m \)

Definition of A:

1. A =: A for prime formulae A
2. \( (A \rightarrow B) =: A \rightarrow B \)
3. \( (\forall x A(x)) =: \forall x (Ex (x) \rightarrow A (x)) \)

For the defined connectives \( \forall, \rightarrow \) we have: \( \forall A =: \forall A, (A \rightarrow B) =: (A \rightarrow B) \).

Used are the abbreviations Ex(r ) =: \( \forall \exists i_1 \exists r_i \), r s s =: \( i_1 \exists r_i \), where r s = (r 1 , ..., r m ), s s = (s 1 , ..., s m ).

The deductive framework for the following proofs is arithmetic \( \mathcal{A} \) (section I, II of the deduction frame). The considerations are also largely intuitionistically valid (without (Tnd)); deviations are marked.
At first some direct consequences of the above definitions.

(2.1) \( r^a_e s \rightarrow s^a_e r \)

(2.2) \( r^a_e s \land s^a_t \rightarrow r^a_t \)

(2.3) \( r^a_e r \leftrightarrow \text{Ex}^a(r), \quad r^a_e s \leftrightarrow \bigwedge_{x,y} (x \equiv y \rightarrow r^a x s y) \)

(2.4) \( r^a_s \land \text{Ex}^a(r) \rightarrow \text{Ex}^a(s) \)

(2.5) \( r^a_s \land \text{Ex}^a(r) \rightarrow r^a_e s \)

(2.6) \( \bigwedge x^a \text{Ex}^a(x), \quad \overline{a}_e \)-definition for all types is deductive equivalent to (E1), \( \overline{a}_e \)-definition for all types.

Proof:

\( \rightarrow: \) With \( \varphi_{u,y} \equiv r(u,y) \) according to (R1), where \( u,y \) are the variables of \( r \).

\( \leftarrow: \) Induction on types with iterated (E1)-application.

(2.7) \( \text{Ex}(\varphi) \land y \equiv y \rightarrow r(y)_e r(y) \) where \( \varphi \) are the functional constants, \( y \) the variables of \( r(y) \)

Proof: Induction on T-rank 1 of \( r \)

I. \( l=0 \): case 1: \( r(y) \equiv \varphi \) conclusion from premise with (2.3)

II. \( l \neq 0 \): \( r(y) \equiv s(y)(t(y)) \)

By induction hypothesis:

(1) \( \text{Ex}(\varphi) \land y \equiv y \rightarrow s(y)_e s(y) \)

(2) \( \text{Ex}(\varphi) \land y \equiv y \rightarrow t(y)_e t(y) \)

We have to show:

(\( \ast_1 \)) \( \text{Ex}(\varphi) \land y \equiv y \rightarrow \text{Ex}(s(y)(t(y))) \land \text{Ex}(s(y)(t(y))) \)

(\( \ast_2 \)) \( \text{Ex}(\varphi) \land y \equiv y \rightarrow \text{Ex}(s(y)(t(y))) \rightarrow s(y)(t(y)) \equiv s(y)(t(y)) \)

Ad (\( \ast_1 \)):

\( \text{Ex}(\varphi) \land y \equiv y \rightarrow \text{Ex}(s(y)) \land \text{Ex}(t(y)) \) \hspace{1cm} (1),(2), def.

\[ \rightarrow s_1(y) \equiv y_1 \rightarrow s(y)(t(y)) \equiv s(y)(t(y)) \] \hspace{1cm} (2.3), def.

\[ \rightarrow \text{Ex}(s(y)(t(y))) \]

Analogous: \( \text{Ex}(\varphi) \land y \equiv y \rightarrow \text{Ex}(s(y)(t(y))) \)