Remarks on the Poisson process.

A. Rényi

The (inhomogeneous) Poisson process on the real line is usually characterised as a stochastic additive set function \( f(E) \) defined for each bounded Borel subset \( E \) of the real line such that

a) the random variable \( f(E) \) has for each bounded Borel set \( E \) a Poisson distribution, i.e.

\[
P(f(E) = n) = \frac{[\lambda(E)]^n \cdot e^{-\lambda(E)}}{n!} \quad (n = 0,1,\ldots)
\]

where \( \lambda(E) \) is a nonatomic measure on the real line such that \( \lambda(E) \) is finite for each finite interval \( E \), and

b) if \( E_1, E_2, \ldots, E_n \) are mutually disjoint bounded Borel sets the random variables \( f(E_1), f(E_2), \ldots, f(E_n) \) are independent.

If we put \( f_t = f([0,t)) \) for \( t > 0 \), this means that \( f_t \) is a process with independent increments such that \( f_t - f_s \) has a Poisson distribution with mean value \( \Lambda(t) - \Lambda(s) \) where \( \Lambda(t) \) is the \( \lambda \)-measure of the interval \( [0,t) \) if \( t > 0 \) and \( -\Lambda(t) \) is the \( \lambda \)-measure of the interval \( [t,0) \) if \( t < 0 \). D. Szász (oral communication) asked the question whether there exists a point process for which a) holds but b) does not hold.

We shall show in this note that such a process does not exist, i.e. the usual supposition about independence in the above characterisation of the Poisson process is unnecessary, as it follows from the Poissonity of the distribution of \( f(E) \); in other words we prove that the supposition b) is a consequence of the supposition a).

More exactly we prove the following

**Theorem 1.**

Let \( J \) denote the family of all subsets of the real line which can be obtained as the union of a finite number of disjoint finite intervals \( [a,b) \) closed to the right and open to the left. Let \( f(E) \) be an additive stochastic set function defined for each \( E \in J \), i.e. such that if \( E_1 \) and \( E_2 \) are disjoint one has \( f(E_1 + E_2) = f(E_1) + f(E_2) \)
Suppose that for each \( E \in \mathcal{F} \), \( \mathbb{P}(E) \) has a Poisson distribution with mean value \( \lambda(E) \) where \( \lambda(E) \) is a nonatomic measure on the Borel subsets of the real line, which is finite for each \( E \in \mathcal{F} \). Then it follows that if \( E_1, \ldots, E_n \) are disjoint sets (\( E_k \in \mathcal{F} \)) the random variables \( \mathbb{P}(E_1), \ldots, \mathbb{P}(E_n) \) are independent, i.e. \( \mathbb{P}(E) \) is a Poisson process.

**Proof of theorem 1.** Let \( A(E) \) denote the event \( \mathbb{P}(E) = 0 \). If \( E \) is the union of the disjoint sets \( E_j \in \mathcal{F} \) (\( j = 1, 2, \ldots, n \)) then (\( \ast \)) clearly \( A(E) = A(E_1) \cdots A(E_n) \) because \( \mathbb{P}(E) = \sum_{j=1}^{n} \mathbb{P}(E_j) \) and thus \( \mathbb{P}(E) = 0 \) iff \( \mathbb{P}(E_j) = 0 \) for \( j = 1, 2, \ldots, n \).

But by supposition

\[
\mathbb{P}(A(E)) = \mathbb{P}(\mathbb{P}(E) = 0) = e^{-\lambda(E)} = \prod_{j=1}^{n} e^{-\lambda(E_j)} = \prod_{j=1}^{n} \mathbb{P}(A(E_j))
\]

Thus it follows that if the sets \( E_1, \ldots, E_n \) are disjoint, the events \( A(E_1), \ldots, A(E_n) \) are independent.

Now let \( \mathbb{I}(A(E)) \) be the indicator of the event \( A(E) \). Let \( E \in \mathcal{F} \) and \( F \in \mathcal{F} \) be two disjoint sets. For any \( \varepsilon > 0 \) we can clearly decompose \( E \) into disjoint intervals \( E_i \) \((1 \leq i \leq n)\) and \( F \) into disjoint intervals such that

\[
\max_{i} \lambda(E_i) < \varepsilon \quad \text{and} \quad \max_{j} \lambda(F_j) < \varepsilon
\]

Now evidently

\[
\mathbb{I}(E) \neq \sum_{i=1}^{n} \mathbb{I}(A(E_i)) \implies \max_{i} \mathbb{P}(E_i) \geq 2
\]

and

\[
\mathbb{I}(F) \neq \sum_{j=1}^{m} \mathbb{I}(A(F_j)) \implies \max_{j} \mathbb{P}(F_j) \geq 2.
\]

On the other hand for any \( B \in \mathcal{F} \)

\[
\mathbb{P}(\mathbb{P}(B) \geq 2) = \sum_{k=2}^{\infty} \frac{\lambda(B)^{k} e^{-\lambda(B)}}{k!} \leq \lambda^{2}(B)
\]

Thus

\[
\mathbb{P}(\mathbb{I}(E) \neq \sum_{i=1}^{n} \mathbb{I}(A(E_i)) \leq \sum_{i=1}^{n} \lambda^{2}(E_i) < \varepsilon \lambda(E)
\]

(\( \ast \)) Here and in what follows the product of events denotes the joint occurrence of these events.