OPTIMAL CODING IN WHITE GAUSSIAN CHANNEL WITH FEEDBACK

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Let us consider the coding scheme for the transmission of a Gaussian random variable $\theta$ through a white Gaussian channel with feedback. The model is formulated as follows: $d\xi_t = z(t, \xi_{s}, \theta) \, dt + \mathcal{N} \, dw_t$, $t \geq 0$. Shiryayev [1] constructed the "optimal" coding $z^*(t)$ in the linear codings:

$$z(t) = z(t, \xi_{s}, \theta) = A_0(t, \xi_{s}) + A_1(t, \theta),$$

conditioned by $\mathbb{E}z^2(t) \leq P_0$, $t \geq 0$ (Theorem 1). The "optimal" means that the coding minimizes the mean square error of estimating $\theta$ based on the observations $\xi_s$, $0 \leq s \leq t$. The purpose of this paper is to show that, in all codings $z$ (not necessarily of linear type) conditioned by $\mathbb{E}z^2(t) \leq P_0$, the coding $z^*$ is optimal in the above sense and moreover $z^*$ is optimal also in the sense of maximizing the information quantity between $\theta$ and $\xi$, $0 < s < t$ (Theorem 2).

1. Let $\theta$ be a Gaussian random variable with the distribution $N(m, \gamma)$, which is the message to be transmitted. And let $w = \{w_t, \, t \geq 0\}$ be a standard Wiener process independent of $\theta$. The model for a white Gaussian noise channel with noise-free feedback is formulated as follows: let $\xi = \{\xi_t, \, t \geq 0\}$ be the output, then

$$d\xi_t = z(t, \xi_{s}, \theta) \, dt + \mathcal{N} \, dw_t, \quad t > 0,$$

$$\xi_{s} = 0$$

(1)

where $\xi_{s}$ stands for the path $\xi_s$, $0 \leq s \leq t$ and $\mathcal{N} > 0$ is a given constant. We discuss only on the codings $z$ which satisfy the following assumptions.

Assumption (a). The equation (1) has the unique solution $\xi = \{\xi_t\}$.

(b). $\mathbb{E}z^2(t) \leq P_0$, for each $t \geq 0$, where $P_0 > 0$ is a constant.

Denote by $Z$ the set of all codings satisfying the assumptions (a) and (b).

Our problem is to find the optimal codings $z^*_1$ and $z^*_2$ in the sense of (I) and of (II), respectively:
(I) Minimizing the mean square filtering error of estimating $\theta$ by the data $\xi_t$.

(II) Maximizing the information quantity $I(\theta, \xi_t)$ between $\theta$ and $\xi_t$.

Denote by $\sigma^2(t)$ the minimum of the mean square errors:

$$\sigma^2(t) = \inf_{\zeta \in Z} E(\theta - \hat{\theta}_t)^2,$$

where $\hat{\theta}_t = \hat{\theta}_t(\zeta) = E[\theta | F^\zeta_t]$ is the best estimate corresponding to $\zeta \in Z$. *)

And denote by $I(t)$ the maximum of the information quantity:

$$I(t) = \sup_{\zeta \in Z} I(\theta, \xi_t).$$

2. Shiryayev [1] found out the optimal coding $z^*(t)$ in the linear codings:

$$z(t) = A_0(t, \xi^*_t) + A_1(t) \theta,$$

in the sense of (I). It is given in the following manner. Let

$$A_1^*(t) = \frac{p_0}{\gamma} \exp \left( \frac{p_0}{2N^2} t \right)$$

and

$$A_0^*(t, \xi^*) = - \hat{\theta}_t^* A_1^*(t),$$

where

$$d\xi^*_t = [- A_1^*(t) \hat{\theta}_t^* + A_1^*(t) \theta] dt + N d\omega_t, \quad \xi^*_0 = 0,$$

and

$$\hat{\theta}_t^* = E[\theta | F^\xi_t].$$

Then the coding $z^*(t)$ is defined by

$$z^*(t) = A_0^*(t, \xi^*_t) + A_1^*(t) \theta.$$

Denote by $Z_0$ the set of codings $z \in Z$ of type (4), and put

$$\sigma^2_0(t) = \inf_{z \in Z_0} E(\theta - \hat{\theta}_t)^2.$$

Then the Shiryayev's result can be stated as follows.

*) $F^\zeta_t$ is the $\sigma$-algebra generated by $\xi_s$, $0 \leq s \leq t$. 
