For some class of analytic flows on the 2-dimensional torus, we shall show that the time mean of a quantity \( f \in \mathcal{C}(k) \) converges to its phase mean \( \mathbb{E}(f) \) with the speed \( 1/n \) in the sense of both G. Birkhoff and J. von Neumann.

We consider a differential equation on the 2-dimensional torus \( \mathbb{M}_2 \): 
\[
\frac{dx}{dt} = F(x,y) \quad \frac{dy}{dt} = G(x,y) \quad (\text{mod 1}),
\]
where \( F(x,y) \) and \( G(x,y) \) are analytic functions on \( \mathbb{M}_2 \) such that 
\[
F^2(x,y) + G^2(x,y) \neq 0.
\]
Let \( T_t \) be the transformation on \( \mathbb{M}_2 \) defined by
\[
T_t(x(0), y(0)) = (x(t), y(t)),
\]
where \((x(t), y(t))\) is a solution of the above equation with an initial condition \((x(0), y(0))\), and let \( \mathcal{B} \) be the topological Borel field in \( \mathbb{M}_2 \). We assume there exists an invariant probability measure \( P \) with an analytic density \( p(x,y) \) with respect to Lebesgue measure \( dx dy \).

Denote the rotation number of \( \mathcal{J} \) by \( \gamma : \gamma = \int_{\mathbb{M}_2} F(\omega) dP(\omega) / \int_{\mathbb{M}_2} G(\omega) dP(\omega) \).

We shall prove the following theorem:

**Theorem.** Let \( \mathcal{J} = (\mathbb{M}_2, \mathcal{B}, P, T_t) \) be the flow defined above, and assume that the rotation number \( \gamma \) of \( \mathcal{J} \) is irrational and moreover there exist positive numbers \( L \) and \( H \) such that
\[
|m + n\gamma| \geq L/\ln^H n \quad \text{for any integers } m \text{ and } n.
\]
Then, if \( k-H > 1 \),

\[
\left| \frac{1}{S} \int_{0}^{S} f(T_t(x,y)) dt - E(f) \right| = O(\frac{1}{S}) \quad \text{a.e. (x,y) } \in M_2
\]

holds for any \( C^{(k)} \)-function \( f \), and moreover if \( k-H > 1/2 \)

\[
\left\| \frac{1}{S} \int_{0}^{S} f(T_t(x,y)) dt - E(f) \right\| = O(\frac{1}{S}),
\]

holds for any \( C^{(k)} \)-function \( f \), where \( \| \cdot \| \) means the \( L^2 \)-norm.

**Proof.** Let \( \mathcal{B} = (M_2, \mathcal{B}, dxdy, S_\mathcal{B}) \) be a flow defined by

\[
\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = \gamma.
\]

It is known [1] and [2] that the flow \( \mathcal{J} \) is isomorphic to the flow \( \mathcal{B} \) by an analytic transformation \( f \) on \( M_2 \):

\[
\mathcal{J} f(x,y) = S_t(x,y)
\]

\[
dP(f(x,y)) = dxdy.
\]

We consider first the flow \( \mathcal{B} \). The flow \( \mathcal{B} \) has the discrete spectrum \( \mu_{n,m} = n + \gamma m \) ( \( n \) and \( m \) run over all integers) and eigenfunctions \( \chi_{n,m}(x,y) = \exp i(nx+my): \)

\[
\chi_{n,m}(S_t(x,y)) = \chi_{n,m}(x,t,y+t\gamma) = e^{it\mu_{n,m}} \chi_{n,m}(x,y).
\]

Since \( \gamma \) is irrational, \( \mathcal{B} \) is ergodic and hence there exists an ergodic automorphism \( S_t \). Putting \( T = S_t \), and \( F(\omega) = \int_{0}^{t} f(S_t^t \omega) dt \), we get

\[
E(F) = \int_{M_2} f(S_t^t \omega) dP(\omega) = \int_{0}^{t} f(S_t^t \omega) dP(\omega) dt = t E(f)
\]

and

\[
\frac{1}{S} \int_{0}^{S} f(S_t^t \omega) dt = \frac{1}{S} \sum_{0}^{S} f(T^j \omega) + \frac{1}{S} \int_{0}^{S} f(S_t^t \omega) dt,
\]

where \( \mathcal{L} \) is the integral part of \( S_t \). We shall show the last term of the right side of the above equation is bounded in \( S \). If it is not bounded, we can find sequences \( A_n \uparrow \infty \) and \( S_n \uparrow \infty \) such that

\[
A_n < \left| \frac{1}{S_n} \int_{S_n} f(S_t^t \omega) dt \right| = \left| \int_{0}^{S_n} f(S_t^t \omega) dt \right|
\]

and hence