In this chapter, \( k \) is a cyclotomic extension of \( \mathbb{Q}_2 \), the rational 2-adic numbers. Let \( B \) be an arbitrary cyclotomic algebra over \( k \). It was shown at the beginning of Chapter 4 that \( B \) may be assumed to be of the form:

\[
B = \left( \beta, L/k \right) = \sum_{\sigma \in G} L u_{\sigma}, \quad (u_{\perp} = 1), \tag{5.1}
\]

\[
L = \mathbb{Q}_p \left( \zeta_{2^n}, \zeta_{f^*}^{-1} \right), \quad q = 2^{r^*}, \quad f^* = f_{k/\mathbb{Q}_2}, \quad f = f_{L/k}, \tag{5.2}
\]

\[
u_{\sigma} u_{\tau} = \beta(\sigma, \tau) u_{\sigma \tau}, \quad u_{\sigma} x = x^q u_{\sigma} \quad (x \in L), \tag{5.3}
\]

\[
\beta(\sigma, \tau) = a(\sigma, \tau) \gamma(\sigma, \tau), \quad a(\sigma, \tau) \in \zeta_f, \quad \gamma(\sigma, \tau) \in \zeta_{f^*}. \tag{5.4}
\]

\[
(\beta, L/k) \sim (a, L/k) \otimes_k (\gamma, L/k), \tag{5.5}
\]

where \( \sigma, \tau \in G = G(L/k) \). We will see that \((a, L/k) \sim k \) (cf. Proposition 5.1). For simplicity, put

\[
r = q^f - 1 = 2^{r^* f} - 1. \tag{5.6}
\]

If \( n \leq 1 \), then \( B \sim k \), because the extension \( L/k \) is unramified and the factor set \( \beta \) consists of roots of unity. Hence we always assume \( n \geq 2 \).

Let \( T_0 \) denote the inertia group of \( L/\mathbb{Q}_2 \). Then

\[
T_0 = \langle \theta \rangle \times \langle \iota \rangle, \quad \theta^{2^{n-2}} = \iota^2 = 1, \tag{5.7}
\]
A Frobenius automorphism \( \xi \) of \( L/Q_2 \) is given by

\[
\zeta_2^n = \zeta_2^n, \quad \zeta_1^n = \zeta_1^n, \quad \zeta_2 = \zeta_2.
\]  
\[5.8\]

We have

\[G(L/Q_2) = \langle \theta \rangle \times \langle 1 \rangle \times \langle \xi \rangle.\]  
\[5.9\]

The subgroups of \( T_0 \) are classified into three types:

(i) \( \langle \theta^\lambda \rangle \times \langle 1 \rangle \), \( (\lambda = 0, 1, \ldots, n-2) \),

(ii) \( \langle \theta^\lambda \rangle \), \( (\lambda = 0, 1, \ldots, n-2) \),

(iii) \( \langle 1 \theta^{2\nu} \rangle \), \( (\nu = 0, 1, \ldots, n-3) \).

(Type (i) is the "non-cyclic" case. If \( n = 2 \), then \( \theta = 1 \), \( T_0 = \langle 1 \rangle \), and hence the type (iii) does not arise.) Let \( T \) denote the inertia group of \( L/k \). Then \( T = T_0 \cap G(L/k) \), so \( T \) belongs to one of the above three types.

We apply Lemma 4.1 to the extension \( L/k \). Put

\[e = e_{L/k}, \quad f = f_{L/k}, \quad z = ef, \quad f^* = f_{k/Q_2}.\]  
\[5.11\]

Denote by \( \Omega \) the unramified extension of \( k \) of degree \( z \) and set \( L' = L \cdot \Omega \). Then

\[L' = Q_2(\zeta_2^n, \xi), \quad r' = 2^{e^* z} - 1, \quad z = f_{L'/k}.\]  
\[5.12\]