REFLECTIONS ON THE LEGITIMATE DECK PROBLEM

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ABSTRACT

We study the following problem: given a collection $H = (H_i|1 \leq i \leq n)$ of $n$ graphs, each on $n-1$ vertices, when does there exist a graph $G$ whose vertex-deleted subgraphs are the members of $H$?

1. LEGITIMATE DECKS

A deck of $n$ cards is a collection $(H_i|1 \leq i \leq n)$ of $n$ graphs, each having $n-1$ vertices. If there exists a graph $G$ with vertex set $\{1, 2, \ldots, n\}$ such that

$$G_i \cong H_i \quad (1 \leq i \leq n)$$

(where $G_i$ denotes the subgraph of $G$ obtained on deleting vertex $i$) the deck $(H_i|1 \leq i \leq n)$ is said to be legitimate, and we call $G$ a generator of the deck. Decks which are not legitimate are, of course, illegitimate. The deck shown in figure 1(a) is legitimate: a generator is displayed in figure 1(b); but the deck of figure 2 is illegitimate, because we see from $H_1$ that every generator is acyclic, and from $H_2$ that no generator can possibly be so.

![Figure 1](image1)

![Figure 2](image2)
A less obvious example of an illegitimate deck is given in figure 3.

Figure 3

In the Reconstruction Conjecture [1], the problem is to show that no deck has more than one generator, up to isomorphism. The **Legitimate Deck Problem**, by contrast, seeks a characterization of those decks having at least one generator (in other words, legitimate decks). It was first mentioned by Harary [7] in 1968, more as an aside to the Reconstruction Conjecture than as a problem of independent interest. However, it does appear to be quite a basic question, having links with much existing graph theory.

This paper surveys the first few tentative steps which have been made towards an understanding of legitimate decks. Our notation and terminology is that of Bondy and Murty [2]. However, all graphs are assumed to be simple. Before proceeding, we make a couple of simple observations, based on a fundamental result in the theory of reconstruction known as Kelly's lemma [9].

Let \( H = (H_i \mid 1 \leq i \leq n) \) be a legitimate deck, \( G \) a generator, and \( F \) any graph with \( v(F) < n \). Then Kelly's lemma gives a formula for the number \( s(F,G) \) of subgraphs of \( G \) which are isomorphic to \( F \) in terms of the deck \( H \):

\[
s(F,G) = \sum_{i=1}^{n} \frac{s(F,H_i)}{n-v(F)}
\]

Since \( s(K_2,G) = e(G) \) and \( e(G) - e(G_1) = d_G(i) \), the number of edges and the degree sequence of any generator of a deck can be determined. The following proposition is now easily established.

**PROPOSITION.** Let \( H \) be a legitimate deck, and let \( G \) be a generator of \( H \). Then

(1) if no two vertex degrees in \( G \) are consecutive integers, \( H \) has a unique generator (up to isomorphism) which can be obtained from any card \( H_i \) by adding a new vertex and joining it to the vertices of \( H_i \) whose degrees do not occur in the degree sequence of \( G \);

(2) if all the cards in \( H \) are isomorphic, the unique generator of \( H \) is vertex-