The concept of a perfect system of difference sets has been introduced in [4] as a mathematical model of the following problem in radio-astronomy: A few movable antennas are used in several successive configurations to measure various spatial frequencies relative to some area of the sky. The distances between antennas determine the frequencies obtained. We do not want to miss any frequency, and want to avoid redundancy (repetition of the same spacing between antennas). For more details, the reader is referred to [4] and [5].

Let $c, m, p_1, \ldots, p_m$ be positive integers, let $S_i = \{x_{i1} < x_{i2} < \ldots < x_{ip_i}\}$, $i = 1, \ldots, m$ be sequences of integers, and let $D_i = \{x_{ij} - x_{ik}, 1 \leq k < j \leq p_i\}$, $i = 1, \ldots, m$ be their difference sets. Then we say that the system $\{D_1, \ldots, D_m\}$ is a perfect system of difference sets (PSDS) starting with $c$ if
\[
\bigcup_{i=1}^{m} D_i = \{c, c+1, \ldots, c + \left\lfloor \frac{p_1+1}{2} \right\rfloor \}. \quad \text{Each set } D_i \text{ is called a component of the PSDS } \{D_1, \ldots, D_m\}. \quad \text{The size of } D_a \text{ is } p_a, \text{ the half-size of } D_a \text{ is } r_a = \left\lfloor \frac{p_a}{2} \right\rfloor \text{ where } \left\lfloor x \right\rfloor \text{ denotes the integer part of a real number } x. \text{ Then } p_a = 2r_a + \delta_a \text{ where } \delta_a = 0 \text{ or } 1 \text{ according to whether } p_a \text{ is even or odd. } \text{This notation will be used throughout the paper. The reader will observe that the size of a component is not the number of its elements; if the size of } D_a \text{ is } p_a \text{ then } D_a \text{ has } 1 + 2 + \ldots + p_a = \frac{1}{2} p_a (2p_a - 1) \text{ elements.}
\]

We will briefly review some earlier results concerning PSDS:

A PSDS is called regular if all its components have the same size. A regular PSDS with $m$ components of size $p$, starting at $c$, will be called an $(m, p, c)$-system. In [4], the existence of $(m, p, 1)$-systems has been related to graceful numberings of certain graphs, and some relations between $m, p, c$, necessary for the existence of an $(m, p, c)$-system, have been obtained. Further existence studies have been carried out in [7]; one of the results obtained here is that, if an $(m, p, c)$-system exists, then $p \leq 4$. Without this result, a lot of time and money could have been spent in efforts aimed at finding $(m, p, c)$-systems with large values of $p$. A generalization of this result to the nonregular case has been obtained in [9]: Every PSDS contains at least one "small" component (a
component of size \( \leq 4 \). This has been further generalized in [2]:

Every PSDS starting at \( c \) \((c \geq 1)\) contains at least \( c \) small components. This follows immediately from the inequality (5) below. Proceeding from the inequality (2) it has been proved in [1] that, in a PSDS with \( m \) components with the half-sizes \( r_1 \leq r_2 \leq \ldots \leq r_m \), it is \( r_m \leq K(\sqrt{m} + 1) \) where \( K \) is a constant, and that the average of half-sizes of the components of any PSDS is bounded by a constant. The first result implies that the number of perfect systems of difference sets starting with a given \( c \), which has a given number \( m \) of components, is finite. Moreover, it follows from the results in [2] that \( c \leq m \). This means that the number of all PSDS with a given number of components and all possible starts \( c \) is finite.

Let us now denote (similarly as in [1, 2])

\[
\begin{align*}
    n &= \frac{1}{2} \sum_{a=1}^{m} (2r_a + \delta_a)(2r_a + \delta_a + 1), \\
    s &= \frac{1}{2} \sum_{a=1}^{m} r_a (3r_a + 2\delta_a + 1), \\
    \ell' &= \frac{1}{2} \sum_{a=1}^{m} r_a (r_a + 1), \quad \ell = n - \ell',
\end{align*}
\]

and let \( S = \{c, c+1, \ldots, c+s-1\} \), \( L = \{c+\ell, \ldots, c+n-1\} \), \( M = \{c+s, \ldots, c+\ell-1\} \). Furthermore, let us put \( x_{j+k-l,a} - x_{j-l,a} = d_{ja}^k \), \( j = 1, \ldots, p_a, k = 1, \ldots, p_a+1-j, a = 1, \ldots, m \). Then the elements of \( D_a \) can be represented in the form of a difference triangle

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array}
\]

The top (bottom) \( r_a \) rows of this triangle will be referred to as its upper (lower) half. Then \( s \) and \( \ell' \) denote the number of elements in the lower (upper) halves of all triangles corresponding to \( \{D_1, \ldots, D_m\} \), and \( n \) denotes the number of all elements in all such triangles. According to Proposition 1.1 in [4] we have

\[
\begin{align*}
    \sum_{j=1}^{r_{a}} d_{ja}^{p_a+1-k} &= \sum_{j=1}^{r_{a}} d_{ja}^{p_a+1-k}, \quad k = 1, 2, \ldots, r_{a}, \quad a = 1, \ldots, m
\end{align*}
\]

Adding over \( k \) and \( a \) we get

\[
\begin{align*}
    \sum_{a=1}^{m} \sum_{k=1}^{r_{a}} d_{ja}^{p_a+1-k} &= \sum_{a=1}^{m} \sum_{k=1}^{r_{a}} d_{ja}^{p_a+1-k} \quad j = 1, \ldots, \ell
\end{align*}
\]