On the new projective planes of R. Figueroa

by

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We define a proper projective plane to be a projective plane whose automorphism group does not fix any point or line. Until recently only 2 types of finite proper projective planes had been known: The classical planes and the Hughes planes constructed by Hughes in 1957 (resp., in the smallest case, by Veblen and Wedderburn in 1907). Recently a very interesting third class has been discovered by Figueroa [1], who obtained a plane of order $q^3$ for each prime power $q$ such that $q \not\equiv 1 \pmod{3}$. We present here a slight modification of Figueroa's construction, which works for all prime powers. Also, we investigate the correlation groups of these planes.

Let $q$ be a prime power, $K$ a field of order $q^3$, $(\mathfrak{F}, \mathfrak{B})$ the classical projective plane over $K$ and $(\mathfrak{E}, \mathfrak{S})$ a subplane of $(\mathfrak{F}, \mathfrak{B})$ of order $q$. Define $\mathfrak{F}_1 = \{ \mathfrak{L} \in \mathfrak{F} \mid \mathfrak{L} \cap \mathfrak{S} = \{i\} \}$ and $\mathfrak{F}_1 = \{ \mathfrak{P} \in \mathfrak{F} \mid [\mathfrak{P}] \cap \mathfrak{S} = \{i\} \}$ for $i = 0, 1$. (Here $[\mathfrak{P}] = \{ \mathfrak{L} \in \mathfrak{F} \mid \mathfrak{P} \subseteq \mathfrak{L} \}$.)

Clearly $\mathfrak{S} = \mathfrak{F} \cup \mathfrak{F}_1 \cup \mathfrak{F}_0$ and $\mathfrak{F} = \mathfrak{F} \cup \mathfrak{F}_1 \cup \mathfrak{F}_0$. There is a group $G = \text{PGL}(3,q)$ of automorphisms of $(\mathfrak{F}, \mathfrak{B})$ fixing $(\mathfrak{S}, \mathfrak{F})$ which is generated by perspectivities. Let $\sigma$ be any permutation of $\mathfrak{F}_0 \cup \mathfrak{F}_0$ interchanging $\mathfrak{F}_0$ and $\mathfrak{F}_0$ such that $X^\sigma = X$ for all $X \in \mathfrak{F}_0 \cup \mathfrak{F}_0$ and $\sigma \in G$.

Lemma. Let $X, Y \in \mathfrak{F}_0$ and $X \parallel Y$. Then $XY \in \mathfrak{F}_1$ if and only if $X^\sigma \parallel Y^\sigma$.

Proof. Assume that $XY \in \mathfrak{F}_1$ and let $XY \cap \mathfrak{S} = \{P\}$ (where $XY$ denotes the line joining $X$ and $Y$). The group $G(P)$ consisting of all perspectivities in $G$ with center $P$ has order $q^2(q-1)$ and acts semiregularly on $XY \cap \mathfrak{F}_0$. Thus $G(P)$ is transitive on $XY \cap \mathfrak{F}_0$, and there exists $\sigma \in G(P)$ such that $X^\sigma = Y$. Let $\sigma$ be the axis of $\alpha$. Then $\mathfrak{S} \cap X^{\mathfrak{S}} = X^{\mathfrak{F}} \cap Y^{\mathfrak{S}}$ and clearly $\mathfrak{S} \cap X^{\mathfrak{S}} \in \mathfrak{F}_1$. The dual argument finishes our proof.

We now introduce the following replacement: Denote

$$\mathfrak{L}^* = (\mathfrak{L} \cap \mathfrak{F}_1) \cup ([\mathfrak{L}^{\mathfrak{S}}] \cap \mathfrak{F}_0)^{\mathfrak{S}}$$

for $\mathfrak{L} \in \mathfrak{F}_0$ and

$$\mathfrak{S}^* = \mathfrak{S} \cup \mathfrak{F}_1 \cup \mathfrak{F}_0^*$$

and consider the incidence geometry $(\mathfrak{F}, \mathfrak{S}^*)$. Clearly

$$|\mathfrak{S}| = q^3 + 1$$

for $\mathfrak{L} \in \mathfrak{F}^*$ and

$$|\mathfrak{L} \in \mathfrak{F}^* | P \subseteq \mathfrak{L} | = q^3 + 1$$

for $P \in \mathfrak{F}$. (*)

Let $\mathfrak{L}, \mathfrak{K} \in \mathfrak{F}_0$ and $\mathfrak{L} \parallel \mathfrak{K}$. Assume at first that there exists
S ∈ ℓ ∩ k ∩ P₀. Then ℓᵐ, kᵐ ∈ Sᵐ⁻¹ so that Sᵐ⁻¹ = ℓᵐ kᵐ and S = (ℓᵐ kᵐ)ᵐ is uniquely determined. Hence |ℓ⁺ ∩ k⁺ ∩ P₀| = 1. As ℓᵐ kᵐ = Sᵐ⁻¹ ∈ P₀, we have ℓ ∩ k ∈ P₀ by the dual of our Lemma, so that ℓ ∩ k ∩ P₁ = ∅ and |ℓ⁺ ∩ k⁺| = 1. Assume now ℓ⁺ ∩ k⁺ ∩ P₀ = ∅. Then ℓᵐ kᵐ ∈ P₁ and, again by our Lemma, ℓ ∩ k ∈ P₁. Thus once more |ℓ⁺ ∩ k⁺| = 1.

Let t ∈ P₁ and k ∈ P₀. Suppose that X, Y ∈ t ∩ ℓ⁺ ∩ P₀ and X ⊥ Y. Then XY ∈ P₁ while Xᵐ⁻¹ ∩ Yᵐ⁻¹ = ℓᵐ ∈ P₀, a contradiction. So |t ∩ ℓ⁺ ∩ P₀| ≤ 1 and therefore |t ∩ ℓ⁺| ≤ 2. Thus we have

**Theorem 1.** Let k ⊥ ℓ. If k, ℓ ∈ ℓ⁺, then |k ∩ ℓ| ≤ 2. If k, ℓ ∈ ℓ⁺ ∩ P₀, then |k ∩ ℓ| = 1.

We now choose P = {⟨(x, x, x)⟩ | x ∈ K\{0}}, where x = xⁿ for x ∈ K. Then G is induced by the group of matrices of the form

\[
\begin{bmatrix}
    a & b & c \\
    c & a & b \\
    b & c & a \\
\end{bmatrix}
\]

and determinant ≠ 0, where a, b, c ∈ K. Let S = ⟨(1, 0, 0)⟩ and s be the line corresponding to the kernel of (1, 0, 0)ᵗ. Then Gₛ is induced by matrices of the form

\[
\begin{bmatrix}
    a \\
    \bar{a} \\
    \bar{a} \\
\end{bmatrix}
\]

for a ∈ K\{0}. In particular [G : Gₛ] = q³(q-1)²(q+1) = |P₀|, so that G is transitive on P₀. Also, Gₛ = Gₛ so that there exists a permutation m such that Sᵐ = s and sᵐ = S.

**Theorem 2.** Assume that Sᵐ = s and sᵐ = S.

a) If ℓ ∈ [S] ∩ P₀ and P ∈ s ∩ P₁, then Pᵐ ∈ P₀.
b) (P, P⁺) is a projective plane.
c) s ∩ s⁺ = s ∩ P₁ ∪ {⟨(0, 0, 1)⟩, ⟨(0, 1, 0)⟩}.

**Proof**
a) Clearly (1, 1, 1)ᵗ ∈ P so that ⟨(0, 1, -1)⟩ ∈ s ∩ P₁. As Gₛ = Gₛ is transitive on s ∩ P₁, we can assume P = ⟨(0, 1, -1)⟩. Because G is transitive on P₀, there exists an element x ∈ G such that sˣ = s. Here X⁻¹ ∈ ℓ⁻¹ = s, so that x⁻¹ is represented by a matrix

\[
X⁻¹ = \begin{bmatrix}
    0 & b & c \\
    c & 0 & b \\
    b & c & 0 \\
\end{bmatrix}
\]