HOMOGENEITY OF $S^2 \times T^2$

W.R. Brakes
Faculty of Mathematics
The Open University
Walton Hall
Milton Keynes

1. INTRODUCTION

It is the purpose of this paper to prove the following pair of theorems:

**THEOREM A** $S^2 \times T^2$ is homogeneous.

**THEOREM B** All orientation-preserving homeomorphisms of $\mathbb{R}^2 \times T^2$ are stable.

An intriguing consequence of these results is the

**COROLLARY** The four-dimensional annulus conjecture is true if either
(a) all orientation-preserving homeomorphisms of $S^2 \times T^2$ are stable, or
(b) $\mathbb{R}^2 \times T^2$ is homogeneous.

$\mathbb{R}^2$ denotes the euclidean plane, $S^2$ the two-sphere and $T^2$ the two-fold torus, the cartesian product of two circles. The n-dimensional annulus conjecture refers to the following statement, where $B^n$ is the unit ball in euclidean n-space, $(\frac{1}{2})B^n$ the concentric ball of radius $\frac{1}{2}$ and $C_1$ denotes closure.

AC(n) If $f: B^n \rightarrow \text{Int} B^n$ is an embedding, then $C_1[B^n - f((\frac{1}{2})B^n)]$ is homeomorphic to $C_1[B^n - (\frac{1}{2})B^n]$.

A manifold $M$ of dimension $n$ is homogeneous (resp. weakly homogeneous) if given any two embeddings $f_1, f_2 : B^n \rightarrow M$ there is a homeomorphism $h$ of $M$ such that $hf_1|((\frac{1}{2})B^n) = f_2|((\frac{1}{2})B^n)$ (resp. $hf_1[(\frac{1}{2})B^n] = f_2[(\frac{1}{2})B^n]$). For convenience all manifolds are assumed to be connected and to have empty boundary. In addition it is clearly necessary in order that a manifold be homogeneous that it should admit a homeomorphism that fixes one point and reverses the local orientation at that point, so this is to be assumed wherever appropriate in the following discussion.

It is easily seen that AC(n) implies that all manifolds of dimension $n$ are weakly homogeneous. Similarly the stable homeomorphism conjecture (stated below in convenient form for use here) implies that all manifolds are homogeneous (cf. proof of Theorem A below).
SHC(n)  If \( f : B^n \to \text{Int } B^n \) is an orientation-preserving embedding then there is an embedding
\[
F : C I [B^n - (\frac{1}{2})B^n] \to B^n
\]
such that
\[
F(x) = f(x) \quad \text{if} \quad x \in (\frac{1}{2})S^{n-1}
\]
and
\[
F(x) = x \quad \text{if} \quad x \in S^{n-1}.
\]
This is equivalent to the more usual statement: 'All orientation-preserving homeomorphisms of \( S^n \) are stable.'

If \( n \neq 4 \), SHC(n) is true (classically, with assistance from Brown-Gluck [BG], if \( n \leq 3 \); due to Kirby, Siebenmann and Wall or Hsiang-Shaneson if \( n \geq 5 \), see [Ki] p.575). So in dimensions other than four all manifolds are homogeneous. In the Corollary here it is actually the 'stronger' SHC(4) rather than AC(4) which is implied, but in fact SHC(4) and AC(4) are equivalent, since SHC(3) is known to be true (see section 9, p.11 of [BG]).

It is known that for some manifolds \( M^n \) it is possible to prove homogeneity independently of SHC(n). For instance, that the \( n \)-sphere \( S^n \) is homogeneous is an immediate consequence of the generalised Schoenflies theorem [Br], and homogeneity of \( S^{n-1} \times S^1 \) is proved in [BG] (Theorem 11.3, p.53). On the other hand, homogeneity of the \( n \)-fold torus \( T^n \) is equivalent to SHC(n). This follows from Kirby's result that all orientation-preserving homeomorphisms of \( T^n \) are stable (Lemma, p.578 of [Ki]) and Lemma 1 below. Theorems A and B serve to pinpoint the borderline between these two situations.

Essentially the same proofs as those given here provide the generalisations of these theorems stated below, without invoking SHC(n).

**Theorem A(n)**
\( S^r \times T^{n-r} \) is homogeneous if \( r \geq 2 \).

**Theorem B(n)**
All orientation-preserving homeomorphisms of \( \mathbb{R}^2 \times T^{n-2} \) are stable.

**Corollary**
Each of the following statements implies SHC(n):
(a) for some \( r \geq 2 \) all orientation-preserving homeomorphisms of \( S^r \times T^{n-r} \) are stable;
(b) \( \mathbb{R}^2 \times T^{n-2} \) is homogeneous.