Abstract. We describe a class of algorithms known as piecewise-linear homotopy methods for solving certain (generalized) zero-finding problems. The global and local convergence properties of these algorithms are discussed. We also outline recent techniques that have been proposed to improve the efficiency of the methods.

1. Introduction

This paper is intended to introduce the reader to a class of algorithms for solving certain (generalized) zero-finding problems. Thus, given a continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, we may wish to find $x^* \in \mathbb{R}^n$ with

$$0 = f(x^*).$$

Since this is such a general and all-embracing problem, it is worthwhile describing some particular instances of (1.1) that motivated the early developers of these methods.

The first algorithm of the class we shall consider was devised by Scarf [69] in 1967. Scarf's method gave a constructive proof of Brouwer's fixed-point theorem, that any continuous function $g$ mapping an $n$-dimensional simplex $S$ into itself has a fixed point. While this is a very general result, Scarf was motivated by the desire to compute an equilibrium price vector for a model of an economy. His algorithm was derived from an earlier method he had developed to prove constructively the non-emptiness of the core in certain $n$-person games--see [68]. The novel and combinatorial proof of convergence was inspired by a similar argument of Lemke and Howson [53] (in a paper that originated the closely related field of complementary pivot theory--see Lemke [52] for a survey). From 1967-1972, those who worked in the field all came from mathematical economics and game theory (Scarf, Hansen, Kuhn and Shapley) or operations research and optimization theory (Eaves, Merrill and Saigal). Their motivation was the computation of fixed points that arise in various economic and game-theoretic models and the computation of...
stationary points in optimization. For instance, if \( g \) maps \( \mathbb{R}^n \) continuously into an \( n \)-simplex \( S \), we may set \( f(x) = x - g(x) \) so that solutions to (1.1) are fixed points of \( g \). If we seek the unconstrained minimizer of a continuously differentiable function \( \theta : \mathbb{R}^n \to \mathbb{R} \), we may set \( f \) equal to the gradient of \( \theta \); then solutions to (1.1) are stationary points of \( \theta \).

Consideration of very natural generalizations in these models--e.g., the introduction of production into economic equilibrium problems or constraints into optimization problems--led to the need to compute solutions to

\[
0 \in F(x^k),
\]

where \( F \) is an upper semi-continuous mapping from \( \mathbb{R}^n \) into nonempty compact convex subsets of \( \mathbb{R}^n \). (This notion of continuity requires that \( \{x \in \mathbb{R}^n : F(x) \subseteq V\} \) be open for each open \( V \subseteq \mathbb{R}^n \).) There is a corresponding fixed-point theorem for such mappings, due to von Neumann and Kakutani, that guarantees the existence of a solution to (1.2) under suitable conditions. Remarkably, the algorithms could easily be adapted to this more general problem.

These general problems and their origins forced certain features on the algorithms devised to solve them. Since it was desired to handle instances of (1.2) as well as of (1.1), no advantage could be taken of smoothness in the basic algorithm. In addition, because the forms of function \( f \) that could arise from economic models was very general, the algorithms had to guarantee convergence for any \( f \) (or \( F \)) satisfying suitable boundary conditions. Initially, these boundary conditions were basically that \( f(x) = x - g(x) \) for some \( g \) mapping a simplex \( S \) into itself. Thus the algorithms were called fixed-point methods. Another term that has been used is simplicial or simplicial approximation methods. It now seems more natural to state the problem in terms of \( f \), and because of the mode of operation of recent algorithms we shall call them piecewise-linear homotopy methods. They are closely related to early embedding methods and more recent continuation methods, as we shall see in section 6. The newer algorithms retain the properties of global convergence under very weak boundary conditions that hold naturally in several applications and of being able to handle point-to-set mappings. In addition,