This paper represents the content of lectures given at the meeting on Universal Algebra and Lattice Theory at Puebla, Mexico in January, 1982. It attempts to survey recent important results in modular lattices, due mainly to Freese, Herrmann, and Huhn, that have a strong geometric content in their ideas and proofs. These results (and others) represent a beautiful amalgamation of the classical results of Birkhoff and von Neumann with the newer disciplines (also due in part to Birkhoff) of universal algebra and model theory. Because the roots of the essential ideas lie in geometry, or perhaps more importantly in the lattice interpretation of projective geometry and the coordinatization thereof, we have attempted to present here a short (in fact too short) introductory course in these basic ideas.

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1. PROJECTIVE GEOMETRIES AS MODULAR LATTICES. In [18], Birkhoff proved that finite dimensional projective geometries could be characterized by their lattice of linearly closed subspaces. This characterization is the fundamental link between modular lattices and projective geometries. In this section we describe that linkage for projective planes and present some related results.

**Definition.** A projective plane is a triple \( (P, L, I) \) where \( P \) and \( L \) are disjoint non-empty sets and \( I \subseteq P \times L \) is a relation satisfying:

- (PP1) For all \( p \neq q \) in \( P \) there exists a unique \( \ell \) in \( L \) such that \( p \in \ell \) and \( q \notin \ell \). We denote this "line" by \( \ell(p, q) \).
- (PP2) For all \( \ell \neq m \) in \( L \) there exists a \( p \) in \( P \) such that \( p \in \ell \) and \( p \notin m \).
  (This "point" is also unique in light of (PP1).)
- (PP3) There exists distinct \( p_1, p_2, p_3, p_4 \) in \( L \) satisfying for distinct \( i, j, k \) in \( \{1, 2, 3, 4\} \), \( p_i \notin \ell(p_j, p_k) \).

We of course call \( P \) the set of points, \( L \) the set of lines, and \( I \) the incidence relation, \( p \) is on \( \ell \). We could also identify each line with the set of

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points incident with the line, namely: \( \hat{\ell} = \{ p \in P: p \parallel \ell \} \), \( \ell \in L \). By defining \( C \subseteq P \) to be linearly closed if \( p, q \in C \) and \( p \neq q \) imply \( \hat{\ell}(p, q) \subseteq C \), we define \( \ell(G) \), the (closure) system of all linearly closed subsets of \( P \). These subsets are precisely \( \{ \emptyset, P \} \cup \{ \{ p \}: p \in P \} \cup \{ \hat{\ell}: \ell \in L \} \) and the Hasse diagram of these subsets looks like:

![Hasse diagram](image)

Alternatively we can let \( M(G) = P \cup L \cup \{ 0, 1 \} \) (assuming these sets are disjoint) and define a partial order relation on \( M(G) \) by:

\[
x \leq y \iff \begin{cases} 
    x = y \\
    x = 0 \\
    y = 1 \\
    x \in P, y \in L \text{ and } x \parallel y
\end{cases}
\]

**PROPOSITION:** \( (M(G); \leq) \) is a lattice in which

(a) For \( p \neq q \) in \( P \), \( p \lor q = \hat{\ell}(p, q) \).

(b) For \( \ell \neq m \) in \( L \), \( \ell \land m \) is the (unique) point guaranteed by (PP2).

We now wish to characterize the lattices \( (M; \lor, \land) \) [or \( (M; \leq) \)] that are produced by projective planes. To do this we need some terminology.

Let \( (M; \lor, \land) \) be a lattice. \( (M; \lor, \land) \) is said to be bounded if there exists \( 0, 1 \in M \) with \( 0 \leq x \leq 1 \) for all \( x \in M \). For \( x \leq y \) in \( M \), \( [x, y] = \{ z \in M: x \leq z \leq y \} \). If \( (M; \lor, \land) \) is bounded, \( a \in M \) (respectively \( c \in M \)) is called an atom (resp. coatom) if \( [0, a] = \{0, a\} \) (resp. \( [c, 1] = \{c, 1\} \)). A spanning 3-frame in a bounded lattice \( (M; \lor, \land) \) is a sequence \( (x_1, x_2, x_3, x_4) \) in \( M \) satisfying (F3.1) \( \lor(p_i; j \neq i) = 1 \), (all \( i \)) and (F3.2) \( p_i \land \lor(p_k; k \neq i, j) = 0 \), (all \( i \neq j \)). Finally a lattice \( (M; \lor, \land) \) is called modular if \( M \) does not contain a sublattice of the form

![Modular lattice](image)

**THEOREM.** Projective planes "are" precisely modular lattices \( (M; \lor, \land) \) that contain a spanning 3-frame \( (p_1, p_2, p_3, p_4) \) of atoms.