FINITE FORBIDDEN LATTICES

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ABSTRACT. Let $L$ be any finite simple lattice of at least three elements, whose co-atoms intersect to 0. One principal result of the paper is that $L$ is not dual isomorphic to the lattice of subvarieties of any locally finite variety. A second principal result is that these statements are equivalent: (i) $L$ is isomorphic to the congruence lattice of a finite algebra with one basic operation; (ii) $L$ is isomorphic either to the subspace lattice of a finite vector space, or for some permutation $\sigma$ of a finite domain, to the lattice of equivalence relations invariant under $\sigma$.

INTRODUCTION. Lattices isomorphic to the congruence lattice of an algebra are called algebraic, and have been characterized abstractly. Interesting questions arise when we ask about representations in special classes of algebras. The simplest questions of this sort, concerning the number and the type of operations required to represent a lattice, have been the source of interesting and serendipitous results, for instance the discovery of innocuous conditions on an algebraic lattice which force every representing algebra to have rather well behaved operations.

"It is a trivial fact that, while representing lattices as congruence lattices of algebras, we can confine ourselves to unary algebras." (Quoted from Pálfy, Pudlák [9].) Indeed, this is true in the sense that, for every algebra, there exists a unary algebra having the same universe and the same congruence lattice. And the quoted authors make good use of this fact. On the other hand, algebras whose operations are unary are not widely important in mathematics. And so a major theme in recent investigations of congruence lattices has been the elucidation of the influence that the "shape" of a congruence lattice can have on the structure of $n$-ary operations, $n \geq 2$, that preserve all the congruences. The discovery that deep influences of this sort exist is a belated but spectacular by-product of work on the simplest of special representation problems, the problem of characterizing abstractly the algebraic lattices that are isomorphic to the congruence lattice of an algebra having just one binary operation.

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In [1], Freese, Lampe, and Taylor prove that for \( L \) the lattice of subspaces of an infinite dimensional vector space over an uncountable field \( F \), if \( A \) is any algebra whose congruence lattice is isomorphic to \( L \), then every operation of \( A \) satisfies a quasi-identity called the "term condition" (which is strongly suggestive of linear operations), and \( A \) must have at least \( \kappa = |F| \) many operations.

(Lampe described, in 1977, a class of lattices that force the term condition, and Freese and Taylor modified his result to include these subspace lattices.) Taylor [14] uses Lampe's discovery to construct a countable algebraic lattice that is not representable as the congruence lattice of any semigroup.

As a consequence of the mentioned result, and another result of Lampe's, the sought-after characterization for congruence lattices of algebras with one binary operation now seems very remote. He proved, in [4], that any algebraic lattice whose unit element is compact (and this is true of every finite lattice) can be represented as the congruence lattice of an \textit{infinite} algebra with one binary operation.

This paper is the outcome of a long-standing interest in the characterization of lattices that can be represented as the congruence lattice of a \textit{finite} algebra with one binary operation. The naive expectation that all finite lattices must admit such a representation is shown here to be false. But our main contribution is to show that the shape of the congruence lattice can force the operations of a finite algebra to satisfy the term condition, and in fact, to satisfy an even stronger quasi-identity. This paper, like [9], also clarifies some of the obstacles which make it so difficult to construct finite algebras with prescribed congruence lattices. At this moment, it remains unknown if every finite lattice is isomorphic to the congruence lattice of a finite algebra. (We shall return to this question a few paragraphs later.)

In 1969, we had shown that for every finite algebra \( A \), there are finite algebras \( B \) and \( C \) with \( \text{Con} \ A \cong \text{Con} \ B \cong \text{Con} \ C \), and such that \( B \) has just four unary operations, and \( C \) has only a binary operation and a unary operation. These proofs can be found in [2; Theorem 4.7.2] and in [5; §§1-2]. It remains unknown today whether the four unary operations can be replaced by three, or by two.

One of the more striking results in this paper is the description of a congruence lattice of two unary operations on a finite set (in fact, of two permutations), which is not isomorphic to the congruence lattice of any finite algebra with one operation (\( n \)-ary, for any \( n \)). Below is pictured such a lattice.

![Diagram](image)

The alternating group on four letters, \( A_4 \), can be represented as a group of 12