Let $R$, $S$ be rings with unit element. By $R$- or $S$-modules we shall always mean unital modules. Let $U = \_R \S$ be an $R$-$S$-bimodule. Then for every left $R$-module $X$ a canonical homomorphism

$$\varphi(X): U \otimes_S \text{Hom}_R(U, X) \longrightarrow X$$

is defined by $\varphi(X)(u \otimes f) = f(u)$ for $u \in U, f \in \text{Hom}_R(U, X)$. Similarly, for every left $S$-module $Y$ a canonical homomorphism

$$\sigma(Y): Y \longrightarrow S \text{Hom}_R(U, U \otimes_S Y)$$

is defined by $(\sigma(Y)y)u = u \otimes y$ for $y \in Y, u \in U$. The natural transformations $\varphi$ and $\sigma$ are fundamental tools in the categorical theory of modules. Indeed, the theory of Morita equivalence is precisely for the case where both $\varphi$ and $\sigma$ are isomorphisms. Generalizing the Morita theory, Fuller [1] considered the case where $\sigma(Y)$ is an isomorphism for all left $S$-modules $Y$ and $\varphi(X)$ is an isomorphism for all $X$ in a certain class of left $R$-modules, and succeeded in obtaining a theorem characterizing the structure of $U$ which corresponds to this case. On the other hand, Sato [5] has recently worked out determining the type of $U$ for which $\sigma(Y)$ is an isomorphism for all left $S$-modules $Y$, and as an application given an improvement and sharpening of Fuller's theorem. In the present note, by observing $\varphi$ rather than $\sigma$, we attempt to get another approach, which, combined with Sato's results, yields a further refinement and clarification of Fuller's characterization.
Let $\overline{X}$ denote, for each left $R$-module $X$, the image of $\rho(X)$ i.e. the sum of all homomorphic images of $RU$ in $X$. Then clearly $\text{Hom}_R(U, \overline{X}) = \text{Hom}_R(U, X)$, and this implies that $\rho(\overline{X})$ is an isomorphism if and only if $\rho(X)$ is a monomorphism. Let $\text{Gen}_R(U)$ be the class of those left $R$-modules $X$ for which $\overline{X} = X$. It follows then that $\rho(X)$ is an isomorphism for all $X$ in $\text{Gen}_R(U)$ if and only if $\rho(X)$ is a monomorphism for all left $R$-modules $X$. Now that $X$ is in $\text{Gen}_R(U)$ means that $X$ is a sum of homomorphic images of $RU$, and this is also equivalent to the condition that $X$ is a homomorphic image of a direct sum of copies of $RU$, that is, there exist an index set $A$ and an epimorphism $RU^A \rightarrow RX$, where $U^A$ means the $A$-times direct sum of $U$. Generally, each homomorphism $h: RU^A \rightarrow RX$ can be identified with a family $\{h_\alpha\}_{\alpha \in A}$ of homomorphisms $h_\alpha: RU \rightarrowRX$ such that $h(\{u_\alpha\}) = \sum h_\alpha(u_\alpha)$ for every $\{u_\alpha\} \in U^A$.

(Here, $u_\alpha = 0$ for all but a finite number of $\alpha$, while $h_\alpha$'s need not satisfy such a condition.)

**Lemma 1.** Let $h = \{h_\alpha\}$ be an epimorphism $RU^A \rightarrow RX$ and let $\rho(X)$ be a monomorphism. Then $U \otimes \text{Hom}_R(U, X) = U \otimes \sum \text{Sh}_\alpha$.

**Proof.** Let $t$ be any element of $U \otimes \text{Hom}_R(U, X)$ and let $x \in X$ be the image of $t$ by $\rho(X)$. Since $h: RU^A \rightarrow RX$ is an epimorphism, there exists $\{u_\alpha\} \in U^A$ such that $x = h(\{u_\alpha\}) = \sum h_\alpha(u_\alpha)$. Consider now $\sum u_\alpha \otimes h_\alpha \in U \otimes \text{Hom}_R(U, X)$. Its image by $\rho(X)$ is also $\sum h_\alpha(u_\alpha) = x$. Since however $\rho(X)$ is a monomorphism, it follows $t = \sum u_\alpha \otimes h_\alpha$, which shows that $U \otimes \text{Hom}_R(U, X) = U \otimes \sum \text{Sh}_\alpha$. 