INJECTIVE QUOTIENT RINGS OF COMMUTATIVE RINGS

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INTRODUCTION

In the broadest sense, this is a study of commutative rings which satisfy the (finitely) pseudo-Frobenius (or (F)PF) condition: All (finitely generated) faithful modules generate the category mod-\( R \) of all \( R \)-modules. These rings include: Prüfer rings, almost maximal valuation rings, self-injective rings, e.g., quasi-Frobenius (QF) and pseudo-Frobenius (PF) rings, and finite products of these. (In fact, any product of commutative FPF rings is FPF [34]; hence, any product of commutative PF rings is FPF (cf. §9).)

If \( R \) is FPF, so is its (classical) ring of quotients \( Q_{cl}(R) \) and its maximal quotient ring \( Q_{max}(R) \). All known FPF rings are (classically) quotient-injective in the sense that \( Q_{cl} \) is injective.\(^2\) We conjecture that all FPF rings are quotient-injective, and prove this in the three cases: (1) local rings (Proposition 7 and Theorem 9B); (2) Noetherian rings (Theorem 11; Endo's Theorem \[25\]); (3) reduced rings (Proposition 3B and Theorem 4). Moreover, any FPF commutative ring \( R \) splits, \( R = R_1 \times R_2 \), where \( R_1 \) is semihereditary, and \( R_2 \) has essential nilradical. (If \( R \) is semilocal or Noetherian, then \( R_2 \) is injective.) Thus any reduced FPF ring has regular injective \( Q_{cl} \), and conversely any quotient-injective semihereditary ring is FPF (Theorem 4).

A ring is pre-FPF iff all (finitely generated) faithful ideals are generators, and we

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\(^2\) In general, \( Q_{cl} \) is injective as an \( R \)-module iff it is a self-injective ring [21].
show this occurs iff all such ideals are actually projective. This is proved via a partial converse of Azumaya's theorem (corollary to Proposition 5A) stating that all faithful finitely generated projectives are generators. The partial converse states that all "rank-1" generators are finitely generated projective. (See Theorem 1C and Propositions 1D and 1F.) This enables us to prove that any FPF ring $R$ has flat epic $Q_{\text{max}}$ (Theorem 1E). A ring $R$ is right Kasch if every simple right module embeds in $R$; equivalently, maximal right ideals have nonzero left annihilators. Clearly, any commutative Kasch ring is pre-PF. Moreover, every pre-PF commutative ring has Kasch $Q_{\text{max}}$ (Proposition 1G).

Noetherian quotient-injective rings have been characterized by Bass [21]:

The zero ideal is unmixed and all of its primary components are irreducible. In the general case, while the problem of characterizing quotient-injective rings is still open, Vámos [19] determined all fractionally self-injective ( = FSI) rings, that is, rings such that every factor ring is quotient-injective (see Theorem 19), and related them to the structure of $\sigma$-cyclic rings, that is, rings over which every finitely generated module is a direct sum of cyclics. It follows easily from the structure theory of Brandal [27], Vámos [19], and the Wiegands [20] that every $\sigma$-cyclic ring is quotient-injective. (See Theorem 19.)

The condition that every factor ring of $R$ is FPF is called CFPF, and is related to Vámos' condition FSI. The truth of our conjecture would imply that $R$ CFPF $\Rightarrow$ FSI. A local ring $R$ is CFPF iff $R$ is an almost maximal valuation ring (Theorem 5B). Thus CFPF $\iff$ FSI for a local ring $R$ by a theorem of Vámos [19]. (These results imply that not every valuation ring (VR) is quotient-injective, since otherwise every factor ring of a VR would be quotient-injective, hence FSI whence almost maximal.) Also CFPF $\Rightarrow$ FSI for Noetherian $R$ (Corollary 12C).

It is shown that a local ring $R$ is FPF iff $Q_{\text{cf}}$ is injective and the zero divisors $P$ is a "waist" of $R$ such that $R/P$ is a valuation ring. (This general-

3 Another characterization: The dual of any finitely generated module is reflexive [21].