WHITEHEAD PRODUCTS AND DIFFERENTIAL FORMS

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This talk served as an introduction for the theorem asserting that the cohomology of the Lie algebra of vector fields on R^n with compact supports is isomorphic to the real cohomology of the n-th loop space of a wedge of spheres (cf.[2]).

1. Definition of Whitehead products.

Let D^n be the set of vectors in R^n of length ≤ 1 and $\partial D^n = S^{n-1}$ its boundary. Let X be a topological space with a base point x_o.

Given two continuous maps $f_i : (D^{p_i}, \partial D^{p_i}) \to (X, x_o)$ $i = 1,2,$ the Whitehead product of f_1 and f_2 is the map $[f_1, f_2]$ of $\partial(D^{p_1} \times D^{p_2})$ in X defined by :

$$[f_1,f_2](x_1,x_2) = \begin{cases} f_1(x_1) & \text{for } x_2 \in \partial D^{p_2} \\ f_2(x_2) & \text{for } x_1 \in \partial D^{p_1} . \end{cases}$$

The f_i represent elements φ_i of the homotopy groups $\pi_{p_i}(X, x_o)$. The homotopy class $[\varphi_1, \varphi_2] \in \pi_{p_1+p_2-1}(X, x_o)$ of $[f_1, f_2]$ depends only on φ_1 and φ_2 and is called the Whitehead product of φ_1 and φ_2.

With the correct grading and the correct sign in front of the Whitehead product, the homotopy groups of X tensored by the reals R form a graded Lie algebra over R.

A graded Lie algebra L is a collection of vector spaces L_q, q an integer, with a bilinear map

$$L_p \times L_q \xrightarrow{[\,,\,]} L_{p+q}$$

such that

$$[x,y] = -(-1)^{\deg x \deg y}[y,x]$$

$$[x,[y,z]] = [[x,y],z] + (-1)^{\deg x \deg y}[y,[x,z]] .$$

For instance one gets such a graded Lie algebra over R by considering $\pi_*(X)\otimes R$ as a graded vector space whose homogeneous component of degree q is $\pi_{q+1}(X)\otimes R$, and defining the bracket of $\varphi_1\otimes 1\in\pi_{p_1}(X)\otimes R$ and $\varphi_2\otimes 1\in\pi_{p_2}(X)\otimes R$ as $(-1)^{p_1}[\varphi_1,\varphi_2]\otimes 1$.

2. Using forms to detect Whitehead products.

Recall that when X is a differentiable manifold, one can use differentiable maps and homotopy to define the homotopy groups.

We begin with a lemma often used for detecting the Hopf invariant.

LEMMA 1. *Let ω_1 and ω_2 be two forms on the differentiable manifold X of degrees p_1 and p_2 greater than one. Assume that*

$$d\omega_1 = d\omega_2 = 0 \quad \text{and} \quad \omega_1\omega_2 = 0 .$$

For a smooth map $f : S^{p_1+p_2-1}\to X$, the form $f^*\omega_1$ is of degree $p_1<p_1+p_2-1$, so there is a form α_1 such that $d\alpha_1 = f^*\omega_1$.

The number

$$h_f = h_f(\omega_1,\omega_2) = \int_{S^{p_1+p_2-1}} \alpha_1 . f^*\omega_2$$

is independent of the choice of α_1 and of f in its homotopy class. It defines a homomorphism of $\pi_{p_1+p_2-1}(X)$ in R .

COROLLARY. If $h_f(\omega_1,\omega_2)\neq 0$, then f represents an element of infinite order in $\pi_{p_1+p_2-1}(X)$.