SEGAL ALGEBRAS AND DENSE IDEALS IN BANACH ALGEBRAS

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I should like very much to thank Professor K.R. Unni, and other members of the Institute, for inviting me to this conference and affording me the opportunity to present this paper. Also, it gives me great pleasure to record here my thanks to Professor Hans Reiter for words of encouragement, some years ago during a time at which I had nearly abandoned mathematics. Furthermore, Professor Reiter's published works are directly responsible for the course of my own investigations.

As a matter of fact the things I have chosen to talk about today have for their origin some results announced by Hans Reiter in 1965 [38].

The notion of Segal algebra, introduced by Professor Reiter in 1965 is now fairly familiar to mathematicians working in modern harmonic analysis. The idea of introducing via axioms classes of Banach algebras that have properties common to a 'well known' Banach algebra is not new. However in many respects, Segal algebras and their generalizations appear to belong to a class all their own. The (historical) origin of Segal algebras is deeply rooted in classical analysis. Indeed, as is the case with many of the seminal ideas of harmonic analysis, Norbert Wiener defined and studied the 'first' Segal algebra [52], [53]. Subsequently, T. Carleman [10] looked at Wiener's algebra (along with a great deal of Wiener's work in general) from a fresh point of view. As noted, and expounded, by Reiter in his book [39], Carleman's 'fresh point of view' permeates the whole of modern Fourier analysis. This particular example, Wiener's algebra, of a Segal algebra, all continuous \( f \) on the line \( \mathbb{R} \) for which

\[
\sum_{q \in \mathbb{Z}} \max_{x \in [q,q+1]} |f(x)|
\]

is finite, has been
studied by a number of mathematicians: [1], [10], [12], [14], [18], [21], [22], [27], [28], [37], [38], [44], [48], [51], [52], [53], [2], to cite a few. Wiener's algebra is evidently a very popular Banach algebra.

There is no question that I. Segal [44, theorem 3.1] recognised a general algebraic structure underlying Wiener's algebra. However, Segal did not exploit this algebraic structure in a systematic fashion. It remained for Reiter [38] (and subsequently [39], [40]) to recognise the true significance of Segal's 'axioms' and their relationship to analysis on locally compact groups. Presumably Reiter was in possession of his ideal theorem (theorem 3 below) independently of the analogous result in [31] where a specialised version of the ideal theorem (for the algebras \( A_p(G) = \{ f \in L^1(G) \mid \hat{f} \in L^p(G) \} \) was presented. The reason for this presumption is because Segal [44] had already observed the 'ideal theorem' for certain closed ideals (in algebras satisfying the hypotheses of theorem 3.1 of [44]) and with some experience it would not have been difficult to 'guess' the general result.

The Segal algebras \( L^1(G) \cap L^p(G) \), \( 1 \leq p < \infty \) (G not necessarily abelian) have been around for some years. Indeed, as early as 1944 K. Iwasawa [23] studied \( L^1(G) \cap L^p(G) \) and proposed that \( L^1(G) \cap L^p(G) \) (with the norm \( \| f \| = \text{MAX} ( \| f \|_1, \| f \|_p ) \)) be called the group algebra of G. (This is in contrast with Segal [43] terming \( L^1(G) \) with a formally adjoined identity the group algebra of G). Iwasawa[23]

\[ \text{1} \] The contents of this paper indicate quite clearly that Iwasawa was in possession of Gelfand's theory as announced in the early 1940's. International situations of these times makes Iwasawa's knowledge of Gelfand's work of some historical interest. In general Iwasawa's paper is of interest both for its mathematical content and facts concerning the early history of group representations and harmonic analysis.