SOLOVAY'S AXIOM AND FUNCTIONAL ANALYSIS

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1. Taking into account the recent progress in the study on the foundation of Mathematics, there are actually three ways to develop Functional Analysis which depend on the personal convictions of each mathematician.

a) Constructive Functional Analysis

This is the core of Functional Analysis we can get by using exclusively the traditional ways of reasoning universally admitted\(^1\); including the axiom of countable choice and, more adequately, the axiom of inductive definition of sequences\(^2\).

The contents of Constructive Functional Analysis is described, for instance, in my book with De Wilde and Schmets "Analyse Fonctionnelle: théorie constructive des espaces à semi-normes" \(^4\). With a minimum of reasonable separability assumptions, Constructive Functional Analysis contains all the essential facts of Functional Analysis, organized in an elegant theory quite efficient for the

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\(^1\) Of course, the admitted axioms may be listed: they have to secure a precise language, the rules of logical reasoning, the theory of sets, the integers and the axiom of inductive definition of sequences.

\(^2\) This axiom states that there is a law which associates to each set of a sequence of sets an element of this set.

It has historically never been seriously contested by the majority of the mathematicians. Anyway, it cannot be dropped without losing the whole classical analysis.

It has some physical meaning which makes it easy to admit: for lack of an explicit choice function, it is possible to consider as such the listing of the successive choices: this listing is in fact unlimited but may be effectively written as far as wanted.

The axiom of inductive definition of sequences is somewhat more powerful than the axiom of choice: here the choices are done in a sequence of sets, each of which being defined by the previous choices.
Constructive Functional Analysis is the firm basis from which new axioms may be added in a controlled way. This can be done at least in two directions and gives rise to the two following branches of Functional Analysis.

b) Functional Analysis with the additional axiom of non-countable choice: it is actually the Classical Functional Analysis.

c) Functional Analysis with the additional axiom of Solovay.

This seems to be a promising new branch of Functional Analysis: let us call it Solovay Functional Analysis.

2. Let me first give some information about the axiom of non-countable choice and the new axiom of Solovay.

A. The classical exposition of Functional Analysis uses liberally the axiom of non-countable choice.

This axiom is used in the following three ways:
- directly: given a non-countable class of sets, there exists a law which associates an element of each set to each set of the class;
- under the equivalent form of Zorn: a set with a partial order relation has a maximal element if any ordered subset has a upper bound;
- indirectly by consequences: theorem of Tychonov, existence of a Hamel base in any linear space, existence of a ultrafilter finer than a given filter.

The quoted axiom is a very old and respectable axiom: it was born in 1904 under the form of Zermelo [9] and improved in power in 1935 in the form of Zorn [10].

It has been often used - and sometimes abused - and its consequences are well-explored.

It secures proofs without separability restrictions of important propositions of Functional Analysis: theorems of Hahn-Banach,