ON SOME NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Introduction.

The intensive development of the theory of monotone operators in reflexive Banach spaces started about a decade ago when it was realized that these mappings form a very powerful tool in discussing variational boundary value problems for nonlinear equations of the form

\[(\mathcal{A}u)(x) \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,u(x),\ldots,D^m u(x)) = f(x) \quad (x \in \Omega \subset \mathbb{R}^N),\]

provided the functions \(A_\alpha\) satisfy a polynomial growth condition in \(u\) and its partial derivatives of order \(\leq m\). In connection with the decomposition of \(\mathcal{A}\) into its top order part \((|\alpha| = m)\) and its lower order terms, the concept of monotonicity was weakened, and various classes of nonlinear operators which we summarize as 'mappings of monotone type' were introduced.

If the \(A_\alpha\)'s do no longer satisfy a polynomial growth condition, one works with operators of monotone type in Orlicz-Sobolev spaces which may not be reflexive. The study of questions arising in that context is in full progress at the moment; for a review of the present state of the theory cf. Gossez [5].

In this paper we propose to investigate nonlinear elliptic problems which lie somewhat between the two extrema mentioned. We first discuss the solvability of boundary value problems for equations of the form

\[(\mathcal{A}u)(x) \equiv (\mathcal{A}_0 u)(x) + p(x,u(x)) = f(x) \quad (x \in \Omega),\]

where \(\mathcal{A}_0\) is a second order linear elliptic differential operator, while \(p : \Omega \times \mathbb{R} \to \mathbb{R}\) is a function on which no growth restriction is imposed. The study of those equations was initiated by Browder [3];
his results were subsequently sharpened by the writer in [7] by the introduction of 'operators of monotone type with respect to two Banach spaces'. Without proof we mention the basic facts of [7] in Sections 1 and 2, but add also various new results. Note that the restriction to equations of second order is mostly pure convenience.

By reduction to the same abstract class of mappings we prove in Section 3 the solvability of some linear elliptic equations subject to nonlinear boundary conditions.

**Section 1. An abstract existence theorem.**

Let $W, V$ be real reflexive separable Banach spaces with norms $\| \cdot \|_W$ and $\| \cdot \|_V$, and assume $W \subset V$, with a continuous injection mapping of $W$ into $V$. Let $W^*, V^*$ be the conjugate spaces of $W, V$, respectively. Then $V^* \subset W^*$ in the sense that if $f/W$ denotes the restriction of the functional $f \in V^*$ to $W$, then $f/W \in W^*$. By $(w, u)$ we denote the duality pairing, either between $w \in V^*, u \in V$, or between $w \in W^*, u \in W$. The symbols '$\longrightarrow$' and '$\rightharpoonup$' denote strong and weak convergence, respectively.

**Definition 1.1.** Let $A$ be a mapping with domain $D(A) : W \subset D(A) \subset V$, and range contained in $W^*$. We say that $A$ is of type $(M)$ with respect to $V, W$, provided

(i) $A$ is continuous from finite-dimensional subspaces of $W$ to the weak topology on $W^*$;

(ii) if $\{v_n\}$ is a sequence in $W$ and $u \in V$, $g \in V^*$ elements such that $v_n \longrightarrow u$ in $V$, $Av_n \longrightarrow g/W$ in $W^*$, and

$$\lim \sup (Av_n, v_n) \leq (g, u),$$

then $u \in D(A)$ and $Au = g/W$. $A$ is quasi-bounded if for any sequence $\{v_n\}$ in $W$ which is bounded in the $V$-norm, and for which $(Av_n, v_n) \leq \text{const} \times \|v_n\|_V \cdot \|v_n\|_W$, the boundedness in $W^*$ of the sequence $\{Av_n\}$ follows.